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# ESTABLISHMENT AND MAINTENANCE OF COMMUNICATIONS SATELLITE SYSTEMS (SINGLE LAUNCHES)

by

R. N. A. Plimmer

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ESTABLISHMENT AND MAINTENANCE OF COMMUNICATIONS  
SATELLITE SYSTEMS (SINGLE LAUNCHES)

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SUMMARY

A theoretical analysis is presented of the number of launches that are required to establish and maintain a satellite communication system. A simple model is used to represent the process in that launches are assumed to take place with a certain a priori probability whilst the satellites have an exponential probability distribution of lifetimes.

Graphs and equations are given for the various probability distributions of the system during the establishment and maintenance phases of the system and the effect of the time interval between launches is also examined by using a Markov chain representation of the system. It is noted that many of the probability distributions are well represented by a Gaussian distribution.

The results quite clearly indicate the manner in which the number of launches, and hence the cost, depend critically upon the launch success probability and the ratio of the launch interval to the satellite mean lifetime.

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## 1 INTRODUCTION

Several civil and military satellite communication systems have been proposed in recent years whereby radio signals for telecommunication purposes are transmitted between ground stations via orbiting satellites. In order to provide a communication channel which is available at any time of the day these systems require several satellites to be continually maintained in orbit; satellites which have failed, as regards communication capabilities, must be replaced by further satellite launchings so as to maintain sufficient satellites in orbit. The number of satellites required in orbit may vary from a minimum of three for a geostationary system to between fifty and a hundred for a random system. Since the running costs for a random system are high, principally on account of the large number of satellites required, such a system seems rather unattractive at the moment. Consequently, as far as this Report is concerned, investigation has been restricted to medium and high altitude station keeping systems where not more than about 20 satellites are required in orbit.

Of the satellites launched, only a certain proportion will be operationally satisfactory, in that they will be injected successfully into the correct orbital position with all their mechanical and electrical systems working. Present day estimates suggest that the probability of such a successful launch may be as low as 30% although no doubt this will be improved with future experience and research on the reliability of components. Even when a fully operating satellite has been launched successfully into orbit, it will eventually fail to function due to deterioration or breakdown of its numerous working components and will have to be replaced by another satellite. Present design of satellite components aims at achieving a mean lifetime of about five years and eventually ten years.

For costing purposes, preplanning of launch vehicle pads, components, and facilities, it is essential to know the likely number of satellite launchings and the possible launching rates which will be required in order to establish and maintain the system in operation. In this Report an attempt has been made to assess these quantities on the basis of a relatively simple model of the system. It is also shown that many of the probability distributions may be well approximated, for practical purposes, by Gaussian distributions, thus enabling rapid estimates of system performance to be made without detailed calculations.

2 MATHEMATICAL MODEL OF THE SYSTEM

Having decided upon a particular satellite communication system, let it be assumed that a total of at least  $N$  orbiting satellites are required in order to provide continuous communications. Thus  $N$  satellites must first be established in their respective orbits and then, when any satellite fails, it must be replaced as quickly as possible by another. Since a satellite cannot be replaced immediately, it is evident that more than  $N$  satellites must be maintained in orbit to obtain a high probability of continuous communication or else, perhaps for economic reasons, an appreciable break in the communication link must be tolerated. One possible method of attempting to achieve a continuous communication system is simply to duplicate the satellites in pairs and switch to the remaining satellite when its neighbour fails. Although a slight break in transmission is still inevitable at switching it will not mean a complete breakdown unless the neighbour has also failed. Even so, such a method would only seem feasible for a stationary system, for example, attempting to maintain six satellites (three pairs) in orbit. For medium altitude systems demanding a greater number of satellites, a more sensible compromise would be to maintain one or two spare satellites in the system and manoeuvre them into position following the failure of a satellite and accepting the inevitable break in communication. The communication time lost by this method would most likely be less than in attempting to replace the lost satellite by a new launching. Rerouting of the communication link following a satellite failure may also be possible if the increased transmission delays could be tolerated.

In order to formulate a mathematical model for representing the mechanism of establishing and maintaining a communication system certain simplifying assumptions have been made here. In the first place, no account has been taken of the particular configuration of the communication system considered, i.e. whether polar, equatorial, stationary, constant ground-track etc, so that a communication system, for the purposes of this Report, simply consists of a number of satellites  $N$  which must be established and maintained in orbit. Secondly, as regards the launching of satellites, the location of the launching pads and time taken to place a satellite into its correct station do not enter into the analysis. In fact the launching system is considered to be capable simply of launching satellites into orbit at a certain rate and thus the number of available launching pads only determines the frequency with which satellites may be launched. Furthermore, no account is taken of the variation of firing rates between different launching pads and the probability of successfully launching an operationally satisfactory satellite into its correct position in orbit is assumed to have a constant value denoted by  $p$ .

Finally since the operational lifetime of a satellite is determined by the correct functioning of numerous pieces of equipment it seems very reasonable to assume that the lifetime of a satellite has a Poisson probability distribution. Thus we will assume that the probability that a satellite fails in the small increment of time  $dt$  between times  $t$  and  $t + dt$  is  $dt/\ell$ , independent of the time  $t$  and where  $\ell$  is the mean lifetime of the satellite. Consequently, the probability that the satellite operates for a time  $t$  and then fails in the following interval  $dt$  is

$$p(t) dt = e^{-t/\ell} \frac{dt}{\ell} \quad (1)$$

and the probability that it functions for at least time  $t$  is

$$P(t) = \int_t^{\infty} e^{-t/\ell} \frac{dt}{\ell} = e^{-t/\ell} \quad (2)$$

It may be noted in passing that this type of probability distribution is particularly convenient mathematically since the initial operational time of a satellite does not enter into the analysis simply because the probability of any functioning satellite failing in the following instant is independent of time. Any other probability distribution would greatly complicate the analysis since the distribution would depend upon the launch time of the satellite.

### 3 ESTABLISHMENT OF A SATELLITE SYSTEM USING HIGH LAUNCHING RATES

#### 3.1 Theoretical probability distribution

Let us assume that the probability of failure of any satellite in the system is negligible over the established phase, or, in other words, that the launching rate is extremely high. In this limiting case, the number of satellites which have to be launched depends solely upon the probability of making a successful launch,  $p$ . Now, the probability  $p_N^e(n, p)$  that  $n$  firings will have to be made before there are  $N$  successful launches is equal to the probability that the last firing is successful times the probability of  $(N-1)$  successes in the previous  $(n-1)$  firings. Hence

$$p_N^e(n; p) = p B_{N-1}(n-1, p) \quad (3)$$

where  $B_N(n, p)$  is the binomial distribution function

$$B_N(n, p) = \binom{n}{N} p^N (1-p)^{n-N} \quad (4)$$

and thus

$$P_N^e(n;p) = \binom{n-1}{N-1} p^N (1-p)^{n-N} \quad (5)$$

is the required probability density. Consequently, the probability of obtaining  $N$  satellites in orbit using no more than  $n$  launchings is simply

$$P_N^e(n;p) = \sum_{r=N}^n \binom{r-1}{N-1} p^N (1-p)^{r-N} \quad (N \geq 1) \quad (6)$$

since each term of this series represents the probability of the event that the  $N$ th success occurs on the  $r$ th firing and all these events ( $r = N, N+1, \dots, n$ ) are mutually exclusive. Equation (6) thus gives the probability distribution function of the number of satellites successfully launched into orbit as a function of the number of firings.

An alternative expression for  $P_N^e(n;p)$  can be found from the following reasoning: in  $n$  firings the probability of  $N$  successful launchings is  $B_N(n,p)$  and, consequently, in  $n$  firings the probability of at least  $N$  successes is

$$P_N^e(n;p) = \sum_{r=N}^n B_r(n,p) = \sum_{r=N}^n \binom{n}{r} p^r (1-p)^{n-r} \quad (7)$$

The equivalence of equations (6) and (7), straightforward to prove algebraically, is evident from the reasoning leading to these expressions since either contains all the possible ways of obtaining  $N$  satellites in orbit without making more than  $n$  attempted launches.

### 3.2 Evaluation of the probability distribution

As they stand, the expressions (6) and (7) are in an extremely inconvenient form for calculation purposes but they may be transformed into alternative tabulated forms. The Taylor expansion of a function  $f(x)$  may be written with remainder in the form

$$f(q+p) = \sum_{r=0}^{N-1} \frac{p^r}{r!} f^{(r)}(q) + \int_0^1 \frac{p^N (1-t)^{N-1}}{(N-1)!} f^{(N)}(q+tp) dt \quad (8)$$

and thus, taking  $f = (q+p)^n$  with  $q = 1-p$ , this becomes

$$1 = \sum_{r=0}^{N-1} \binom{n}{r} p^r (1-p)^{n-r} + \int_0^1 \frac{n!}{(N-1)! (n-N)!} p^N (1-t)^{N-1} [1 - p(1-t)]^{n-N} dt .$$

Changing the variable of integration to

$$Z = p(1-t)$$

we obtain

$$1 = \sum_{r=0}^{N-1} \binom{n}{r} p^r (1-p)^{n-r} + \frac{n!}{(N-1)! (n-N)!} \int_0^p Z^{N-1} (1-Z)^{n-N} dZ \quad (9)$$

in which one recognises the integral representation of the incomplete Beta function in the last term, namely

$$B_p(\alpha, \beta) = \int_0^p Z^{\alpha-1} (1-Z)^{\beta-1} dZ . \quad (10)$$

But from equation (7) we have

$$1 - P_N^e(n; p) = \sum_{r=0}^{N-1} \binom{n}{r} p^r (1-p)^{n-r}$$

and thus, by comparison with equations (9) and (10)

$$\begin{aligned} P_N^e(n; p) &= \frac{n!}{(N-1)! (n-N)!} B_p(N, n-N+1) \\ &= \frac{B_p(N, n-N+1)}{B(N, n-N+1)} \equiv I_p(N, n-N+1) \end{aligned} \quad (11)$$

where  $B(\alpha, \beta) \equiv B_1(\alpha, \beta)$  is the usual complete Beta function, defined for integer  $\alpha$  and  $\beta$  as

$$B(\alpha, \beta) = \frac{(\alpha+\beta-1)!}{(\alpha-1)! (\beta-1)!} . \quad (12)$$



The quantity  $I_p(\alpha, \beta)$  has been tabulated by Pearson<sup>1</sup> for the range of variables  $0 \leq p(0.01) \leq 1$  and  $\alpha, \beta \leq 50$ . Using these tables  $P_N^e(n; p)$  has been plotted in Figs. 1 to 6 for  $N = 3, 6, 9, 12, 15$  and  $20$  respectively and for  $p$  ranging from  $0.3$  to  $1.0$  in steps of  $0.1$ . For  $p = 1$ , of course, the distribution is simply a step function at  $n = N$ .

These curves show quite clearly the sharp increase in the number of launches required when the probability of a successful launch is low even when no satellites fail between launches.

### 3.3 Gaussian distribution approximation to $P_N^e(n; p)$

It is shown in Appendix A that for  $N$  large the establishment distribution  $P_N^e(n; p)$  tends towards a Gaussian distribution with a mean  $m = N/p$  and variance  $\sigma^2 = N(1-p)/p^2$ . Thus, if it is assumed that the distribution is Gaussian, the levels of probability for  $n$  are:

$$\begin{aligned} P(n \leq m \pm 4\sigma) &= 50 \pm 49.99\% \\ P(n \leq m \pm 3\sigma) &= 50 \pm 49.85\% \\ P(n \leq m \pm 2\sigma) &= 50 \pm 47.75\% \\ P(n \leq m \pm \sigma) &= 50 \pm 34.15\% \\ P(n \leq m) &= 50\% \end{aligned} \quad (13)$$

Using the Gaussian distribution approximation to  $P_N^e(n; p)$  the corresponding curves have also been plotted for comparison with the exact curves in Figs. 1 to 6, for  $p = 0.5$ . From these it is evident that (particularly for  $N$  greater than about 10) the approximation is extremely good for practical purposes (maximum error for a given probability is about one launching). The main discrepancy between the true and approximate distributions occurs in the 50% probability region, whereas for high degrees of probability the agreement is fairly good, even for small values of  $N$ . Consequently, as a useful working guide, it appears that the distribution may be adequately represented by (particularly for high levels of probability) the Gaussian distribution.

Assuming a Gaussian distribution approximation, Fig. 7 shows the number of launches required to successfully establish  $N$  ( $\leq 30$ ) satellites in orbit with probabilities of 84, 98 and 99.8% and with the probability of a successful launch ranging from  $p = 0.3$  to 1.

#### 4. MAINTENANCE OF THE SYSTEM ASSUMING IMMEDIATE REPLACEMENT OF FAILED SATELLITES

##### 4.1 Without launch failures

Having established a system requiring  $N$  satellites in orbit, let us now consider the maintenance of the system when it is assumed that a failed satellite may be replaced immediately by a new one and, furthermore, that the new launching is perfect. The effect of unsuccessful launchings will be considered later in this section. Thus, under these ideal conditions, the number of satellites in orbit is always maintained constant and we wish to determine the distribution of the firing rate demanded to achieve this. This assumption is equivalent to assuming that the time to establish a satellite in orbit and also the time between possible successful firings are negligible compared with satellite's mean lifetime.

Now, for a single satellite, since the lifetime has a Poisson distribution, the probability that the satellite will be replaced exactly  $k$  times (immediate replacement) during a period  $T$  is

$$p(k; \alpha) = \frac{\alpha^k}{k!} e^{-\alpha} \quad (14)$$

where  $\alpha = T/\ell$ . This is a well known formula representing, for example, the number of changes of state for the 'telegraph signal'<sup>2</sup>. But the sum of two independent quantities having the Poisson distributions  $p(k_1; \alpha_1)$  and  $p(k_2; \alpha_2)$  also has the Poisson distribution

$$p(k_1 + k_2; \alpha_1 + \alpha_2) .$$

Consequently, the probability that  $n$  replacements will be necessary to maintain  $N$  satellites in orbit for a period  $T$  is

$$p(n; \alpha N) = \frac{(\alpha N)^n}{n!} e^{-\alpha N} \quad (15)$$

which, of course, is the same distribution as that required for maintaining a single satellite in orbit for a period  $NT$ . The probability distribution for the maintenance of  $N$  satellites in orbit with  $n$  launchings during a period  $T$ , assuming immediate successful replacement, is therefore

$$P_N^m(n; p=1) = \sum_{r=0}^n p(r; \alpha N) = \sum_{r=0}^n \frac{(\alpha N)^r}{r!} e^{-\alpha N} \quad (\alpha = T/\ell) . \quad (16)$$

#### 4.2 Allowing for launch failures

Let us now assume that the launchings only have a probability  $p$  of being successful but that the failed satellites may still be immediately replaced. Thus the assumptions of para 4.1 still hold apart from the proviso that several instantaneous launchings may have to be made before a successful replacement is accomplished. As before, the probability that  $k$  satellites must be replaced is  $p(k; \alpha N)$  given by equation (15). But the probability that  $k$  satellites will be successfully launched into orbit when not more than  $n$  are launched with success probability  $p$  is  $P_k^e(n; p)$  given by equation (6) or (7). Consequently, the probability distribution for maintaining  $N$  satellites in orbit with  $n$  launchings over a period  $T$  when it is assumed that there is immediate replacement with successful launching probability  $p$  is simply

$$P_N^m(n; p) = \sum_{k=0}^n P_k^e(n; p) p(k; \alpha N) \quad (17)$$

or, explicitly, using equations (7) and (15)

$$P_N^m(n; p) = \sum_{k=0}^n \sum_{r=k}^n \binom{n}{r} p^r (1-p)^{n-r} \frac{(\alpha N)^k}{k!} e^{-\alpha N} \quad (18)$$

Alternatively, equation (6) may be used instead of equation (7) giving

$$P_N^m(n; p) = \sum_{k=0}^n \sum_{r=k}^n \binom{r-1}{k-1} p^k (1-p)^{r-k} \frac{(\alpha N)^k}{k!} e^{-\alpha N}, \quad (19)$$

the individual terms of this series now being independent of  $n$ .

#### 4.3 Evaluation of the probability distribution $P_N^m(n; p)$

As we have seen in section 4.1, for  $p = 1$ , the distribution  $P_N^m$  is determined by the sum of a series of Poisson terms. Several tables of the function  $\frac{a^x e^{-a}}{x!}$  exist of which the most comprehensive appear to be those compiled by Molina<sup>3</sup> of Bell Telephone Laboratories; these tabulate the function for  $a \leq 100$  besides giving values of the sum

$$P(c, a) \equiv \sum_{x=c}^{\infty} \frac{a^x e^{-a}}{x!} ; P(0, a) = 1 . \quad (20)$$

Thus, comparing equations (16) and (20), we have

$$P_N^m(n; 1) = 1 - P(n+1, NT/\ell) \quad (21)$$

and consequently there is no difficulty in plotting the distribution.

The function  $P(c, a)$  may also be written in terms of the incomplete Gamma function, since, if we put  $f(x) \equiv e^x$ ,  $q = -a$  and  $p = a$  in equation (8), it follows that

$$1 = \sum_{r=0}^{N-1} \frac{a^r}{r!} e^{-a} + \int_0^1 \frac{a^N (1-t)^{N-1} e^{-a(1-t)}}{(N-1)!} dt$$

i.e.

$$1 = \sum_{r=0}^{N-1} \frac{a^r}{r!} e^{-a} + \frac{\Gamma_a(N)}{\Gamma(N)} \quad (22)$$

where  $\Gamma(N)$  is the complete Gamma function and  $\Gamma_a(N)$  is the incomplete Gamma function defined by

$$\begin{aligned} \Gamma_a(N) &= a^N \int_0^1 e^{-a(1-t)} (1-t)^{N-1} dt \\ &= \int_0^a e^{-t} t^{N-1} dt . \end{aligned} \quad (23)$$

Consequently, from equations (21) and (22), it follows that the desired relationship is

$$\begin{aligned}
 P(N, a) &= \frac{\Gamma_a(N)}{\Gamma(N)} \\
 &\equiv I\left(\frac{a}{\sqrt{N}}, N-1\right)
 \end{aligned} \tag{24}$$

in the notation of Pearson's<sup>4</sup> tables of the incomplete Gamma function and hence, using equations (21) and (24)

$$P_N^m(n;1) = 1 - I\left(\frac{NT}{\ell \sqrt{n}}, n\right). \tag{25}$$

For  $p \neq 1$ , the distribution is more difficult to calculate since, as is seen by inspection of equations (18) or (19), it now consists of a coupling between the binomial and Poisson distributions. Although  $P_N^m(n;p)$  may be expressed in terms of tabulated quantities it is an extremely tedious process to carry out the calculations involved. For instance,  $P_N^m(n;p)$  can be calculated from the expression

$$P_N^m(n;p) = \sum_{k=0}^n I_p(k, n-k+1) \frac{(aN)^k}{k!} e^{-aN}, \tag{26}$$

obtained by substituting equation (11) in (17).

Rather than perform this calculation for many values of  $n$  and  $p$  a simplification was attempted. Inverting the order of summation in equation (18) we have

$$\begin{aligned}
 P_N^m(n;p) &= \sum_{r=0}^n \sum_{k=0}^r \binom{n}{r} p^r (1-p)^{n-r} \frac{(aN)^k}{k!} e^{-aN} \\
 &= 1 - \sum_{r=0}^n \binom{n}{r} p^r (1-p)^{n-r} \sum_{k=r+1}^{\infty} \frac{(aN)^k}{k!} e^{-aN} \\
 &= 1 - \sum_{r=0}^n \binom{n}{r} p^r (1-p)^{n-r} P(r+1, aN).
 \end{aligned} \tag{27}$$

But, for  $n$  large, the binomial distribution  $B_r(n, p)$  becomes Gaussian<sup>5</sup> with mean  $np$  and variance  $np(1-p)$ . Consequently, equation (27) may be approximated by

$$P_N^m(n; p) \simeq 1 - \sum_{r=0}^n G_r P(r+1, \alpha N) \quad (28)$$

where  $G_r$  is the Gaussian probability density

$$G_r = \frac{1}{\sqrt{2\pi np(1-p)}} \exp \left\{ -\frac{(r-np)^2}{2np(1-p)} \right\} \quad (29)$$

In this form, equation (28) takes into account the 'tail' of the Poisson distribution for large values of  $n$ . The quantity  $P(r, \alpha N)$  is unity for  $r = 0$  (equation (20)), approximately  $\frac{1}{2}$  for  $r$  equal to the mean value  $\alpha N$  of the distribution and tends monotonically to zero for  $r$  large. Consequently, for large values of  $n$  the Gaussian distribution  $G_r$ , with mean  $np$ , lies on the 'tail' of  $P(r, \alpha N)$  and hence one would expect equation (28) to give a fairly good representation of the distribution  $P_N^m(n; p)$  for  $n$  large.

Figs. 8, 9 and 10 for  $N = 6, 9$  and  $12$  were constructed using this approximation for  $P_N^m(n; p)$ . The curves correspond to a range of values of  $p$  from  $0.5$  to  $1.0$  and the ratio of the system lifetime  $T$  to the mean lifetime  $\ell$  of each satellite has been taken equal to  $4$ . For instance, for a mean lifetime per satellite of  $5$  years the curves indicate the probability distribution of the number of launches required over a period of  $20$  years in order to maintain the system. For  $p = 1$  the curves were obtained from tabulated values of  $P(c, a)$  directly. Calculations involving the exact formula for  $P_N^m(n; p)$  at isolated points gave extremely good agreement with the values obtained using the approximate formula.

#### 4.4 Gaussian distribution approximation to $P_N^m(n, p)$

It is well known<sup>5</sup> that the Poisson frequency distribution

$$\frac{(\alpha N)^j}{j!} e^{-\alpha N} \quad (j = 0, 1, 2, \dots)$$

tends to the Gaussian distribution for large values of  $\alpha N$ . Consequently, one would expect that the distribution of  $P_N^m$ , being a combination of binomial and Poisson frequency distributions should also become Gaussian for  $\alpha N$  large. That this is the case is proved in Appendix A where it is shown that the limiting Gaussian distribution has a mean value

$$m = \frac{\alpha N}{p} \quad (30)$$

and a variance given by

$$\sigma^2 = \frac{\alpha N(2-p)}{p^2} \quad (31)$$

The distribution curves for  $p = 0.5$  when it is assumed that  $P_N^m$  is normal with the above mean and variance are shown in Figs. 8, 9 and 10 for comparison with the true distribution. It is evident from these that the normality assumption is quite valid for practical purposes where one really only requires an estimate of the distribution curve. As expected the agreement is better for large values of  $\alpha N$ . Fig. 11 also shows that, even for the low value of  $\alpha N = 4$  and  $p = 1$ , the discrepancy between the Gaussian approximation and the exact distribution is generally small and for high levels of probability,  $P_N^m \sim 1$ , extremely good agreement is obtained.

Accepting the validity of the Gaussian approximation the number of satellites that may be required in order to maintain a system over a period of time with a given degree of reliability may easily be constructed. A typical example of this is shown in Fig. 12 for  $T/\ell = 4$ , for systems requiring up to 30 satellites in orbit and  $p$  ranging from 0.3 to 1; the curves have been plotted for 1, 2 and 3  $\sigma$  probability levels of being able to successfully maintain the system.

## 5 EFFECT OF FINITE LAUNCH INTERVAL

One of the shortcomings of the analysis carried out so far is the assumption that any failed satellite may be replaced immediately in the maintenance phase and that during the establishment phase the launchings take place at infinite rate until the system has its correct complement of satellites. Although these simplifying assumptions are acceptable under certain conditions it is of interest to know the consequences when they are not even approximately true; for example, when there are only a small number of launch pads and the time between firings has to be fairly large.

In order to estimate this effect another model of the system has been considered, differing from the previous model in that launchings may only be made at intervals  $\tau$  both during the establishment phase and the maintenance phase. Thus during the establishment phase an attempted launch is made at every launch interval until the required number of satellites are in orbit. During the maintenance phase, if the number of satellites in orbit is less than the required number at the time of a possible launch then a launch is attempted; otherwise no

action is taken until the next possible launch time when a similar decision to launch is made. Since the lifetime of the satellites has been assumed to have a Poisson distribution this means that the probable state of the system at any firing instant depends only upon the state of the system at the previous possible firing instant. From a mathematical point of view this means that the system may be represented by a Markov chain with a transition matrix  $P$  depending upon the possible changes of state of the system during the launch interval  $\tau$ . Appendix B gives the relevant properties of Markov chains which are required in the following analysis. If the  $i$ th state of the system is defined as that corresponding to  $i$  satellites in orbit immediately after a firing then the transition matrix  $P$  will be of order  $(N+1)$ , assuming the maximum number of satellites in orbit to be  $N$ , with elements  $p_{ij}$  corresponding to the state transition  $i$  to  $j$ .

In general there are two ways in which the transition from state  $i$  to  $j$  may occur during the time interval  $\tau$ : (i) either there can be a failure of  $(i-j)$  satellites and an unsuccessful launch, or (ii) a failure of  $i-j+1$  satellites followed by a successful satellite launching. For cases in which  $j$  is greater than  $i+1$ , the transition probability is zero, and for those cases in which  $j$  is equal to  $i+1$  only the transition of type (ii) is possible. Also, if there are  $N$  satellites in orbit at the time of a possible firing, no new launch is attempted. Consequently, if  $f = 1 - e^{-\tau/\ell}$  is the probability that a single satellite will fail during the interval  $\tau$ , the elements of the transition probability matrix are:-

$$\begin{aligned}
 p_{ij} &= q \binom{i}{j} f^{i-j} s^j + p \binom{i}{j-1} f^{i-j+1} s^{j-1} \quad (1 \leq j < i \leq N \text{ or } 1 \leq i = j < N) \\
 &= s^N + N p f s^{N-1} \quad (i = j = N) \\
 &= p s^i \quad (i + 1 = j \leq N) \\
 &= q f^i \quad (j = 0, \text{ any } i) \\
 &= 0 \quad (j > i + 1, \text{ any } i)
 \end{aligned} \tag{32}$$

where  $s = 1 - f = e^{-\tau/\ell}$  and  $q = 1 - p$  so that  $P$  is of the form



$$\begin{array}{c|cccccc}
 & j & 0 & 1 & 2 & 3 & \dots & N \\
 P = & i & 0 & 1 & 2 & 3 & \dots & N \\
 \hline
 0 & q & p & 0 & 0 & \dots & 0 \\
 1 & qf & qs+pf & ps & 0 & \dots & 0 \\
 2 & qf^2 & 2qsf+pf^2 & qs^2+2psf & ps^2 & \dots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & ps^{N-1} \\
 N & qf^N & Nqsf^{N-1}+pf^N & \dots & \dots & \dots & s^N+Npfs^{N-1}
 \end{array} \quad (33)$$

This matrix may also be factorized into the product of two matrices in the form

$$P = DB \quad (34)$$

where the elements of the matrices D and B are

$$\begin{aligned}
 D_{ij} &= 0 & (j > i) \\
 &= \binom{i}{j} f^{i-j} s^j & (j \leq i)
 \end{aligned} \quad (35)$$

and

$$\begin{aligned}
 B_{ij} &= q \delta_{i,j} + p \delta_{i,j-1} & (i \neq N) \\
 &= \delta_{i,j} & (i = N)
 \end{aligned} \quad (36)$$

$$\begin{aligned}
 \text{where } \delta_{i,j} &= 0 & i \neq j \\
 &= 1 & i = j
 \end{aligned}$$

In computing, P may conveniently be obtained by this matrix multiplication since the elements of the  $i$ th row of D are the terms corresponding to the binomial expansion of  $(f+s)^i$  and consequently the elements of successive rows of D may be obtained by multiplying the previous row by  $(f+s)$ . Also successive rows of the matrix B may be obtained by single shifts of the elements of the first row  $(q, p, 0, \dots, 0)$ .

It may also be noted in passing that the factorization  $P = DB$  gives a matrix D depending only upon the probability of failure  $f$  and thus corresponds to the 'death' process of the system whereas B only depends upon the probability of replacement  $p$  and thus represents the 'birth' process of the system in statistical language.

Knowing the basic transition matrix, the probabilities of having various numbers of satellites in orbit as a function of time may easily be obtained by repeated multiplication of  $P$  by itself, since the probability of having  $j$  satellites in orbit after time  $nt$  starting with  $i$  in orbit is given by

$$P_{ij}^{(n)} = [P^n]_{ij} . \quad (37)$$

A typical example of these probabilities is shown in Fig.13 which has been computed for the case  $N = 12$ ,  $p = 0.5$  and  $f = 0.02$  i.e. for a mean satellite lifetime of 5 years this would correspond to a launching rate of 10/year since  $f \triangleq \tau/\ell$  for  $f \ll 1$ . For purposes of comparison, the probabilities of establishment for infinite firing rates ( $f = 0$ ) are also shown in the same diagram for  $p = 0.5$  and  $0.7$ . Quite clearly, the firing rate has a significant effect on the establishment time. For instance, in the case of  $p = 0.5$ , the finite firing rate of 10/year compared with the infinite rate increases by 50% the number of launches required to establish the system with 95% probability.

Since the computation required by equation (37) is extremely time consuming on a digital computer and does not give any detailed statistics of the system an alternative approach was employed. We have already seen that the probability distributions appear to be well represented by a Gaussian distribution and consequently, if we make the assumption that the distribution is approximately Gaussian then the determination of the mean and variance of the Markov process should give a good indication of the distribution. Fortunately, the mean and variance, besides certain other statistics, may be obtained by fairly rapid procedures from the transition matrix. A summary of the relevant formulae is given in Appendix B.

In order to determine the mean and variance of the number of launchings for the establishment of the system, one approach is to make the state  $N$  into an absorbing state by putting the elements  $p_{Ni}$  of the transition matrix equal to zero for  $i \neq N$  and  $p_{NN} = 1$  i.e. we ignore what happens to the system once it has been established. This modified transition matrix then generates an absorbing Markov chain for which we require the mean and variance of the number of steps from state 0 to reach the absorbing state  $N$ . Equations (B6) and (B7) of Appendix B provide this relevant information and have been programmed for a digital computer using the fundamental matrix  $N$  defined by equation (B2).

An alternative method, which also provides additional useful information, is to use the formulae given in Appendix B for the regular Markov chain

generated by the original transition matrix  $P$ . Here the fundamental matrix  $Z$  is defined by equation (B9) and is used in equation (B10) to give the matrix  $M$  with elements  $m_{ij}$ ,  $m_{ij}$  being the mean number of firings starting in state  $i$  before reaching the state  $j$  for the first time. In particular  $m_{0j}$ , is the mean number of firings required to establish  $j$  satellites and clearly

$$m_{ij} = m_{ik} + m_{kj}$$

for  $k$  lying between  $i$  and  $j$ . The variance of the number of firings for establishment is given by equation (B12). It is also worth noting that if  $(N+1)$  is the order of the matrix  $P$  then the mean and variance of the required number of firings for given  $f$  and  $p$ , is obtained for the establishment of all systems requiring no more than  $N$  satellites in orbit. Finally, the elements  $(a_0, a_1, \dots, a_N)$  of the row vector  $\alpha$  which is the solution of  $\alpha P = \alpha$ , represent the fraction of the time that the satellite system can be expected to be in the state of  $0, 1, \dots$  or  $N$  satellites in orbit over a long period of time. Consequently, this provides information regarding the likely state of the system over the maintenance phase.

A computer programme has been written to calculate these various quantities. Although only the case of single launchings has so far been investigated the programme was written for the more general case of multiple launchings by appropriate modification to the transition matrix  $P$ .

Figs.15, 16 and 17 show the expected number of launches which would have to be made to establish systems of 6, 12 and 20 satellites with probabilities 50, 84, 98 and 99.8% as a function of the launch probability  $p$  and the launch interval parameter  $f \triangleq \tau/\ell$ , assuming the Gaussian distribution. (For reference purposes the quantity  $f$  has been plotted in Fig.14 as a function of the launch interval for various mean satellite lifetimes.) It is clear from these diagrams that the launch rate must be kept quite high, even for systems requiring a few satellites, if the probability of successfully launching a satellite is not high, otherwise a very large number of launchings would be required. This is simply due to the fact that there is a high chance of a satellite failing during the mean interval  $1/p$  between successful launchings. For instance, for a system consisting of 6 satellites with mean lifetimes of 5 years, if the launch interval is 3 months i.e.  $f = 0.05$  and the probability of a successful launch is 0.5, the expected time between successful launchings is  $3/0.5 = 6$  months. Consequently the probability of at least one of the satellites failing during this time is  $P_f = 1 - (1-f')^6$  where  $f' = f/0.5 = 0.1$  so that  $P_f = 47\%$  and it

is not surprising that 40 launches i.e. 10 years (Fig.15) are needed to establish the system with 99.8% confidence. On the other hand, if the firing interval is halved, i.e.  $1\frac{1}{2}$  months and  $f = 0.025$ , 30 launchings over nearly 4 years are now required.

The only real remedy for establishing a system, for a given firing rate, in a reasonably short time is to increase the reliability of the launch or the mean lifetime of the satellites. For instance, if  $p = 0.7$  in the last example above then the number of launches is reduced from 30 to about 18 over 2 years. But if the launch rate is high, i.e.  $f$  small, there is little to be gained, at least for the establishment phase, by increasing the mean lifetime, as is indicated by the steep slope of the curves in Figs.15, 16 and 17. This is as expected since there would now be very few satellite failures during the establishment period. In fact, although the number of launches must be greater for  $f \neq 0$ , the effect, as may be seen from the curves in Fig.15, 16 and 17, only becomes really significant as  $f$  is increased above a certain value. On the other hand, the main disadvantage of increasing  $\tau$ , whilst retaining  $f$  constant by increasing the satellite lifetime, is that the establishment time is increased and this is usually of prime importance.

For instance, let us assume that a system of 12 satellites is required in orbit, the mean lifetime of each satellite being 5 years. If there are 4 launch pads, each capable of firing a satellite every 3 months with a success probability of 0.7 ( $\tau = 1/16$  year,  $f = 0.0125$ ) then (Fig.16) the expected number of firings would be about 26 over a period of  $1\frac{1}{2}$  years in order to establish the system with a probability of 98%. Increasing the lifetime to 10 years ( $f = 0.00625$ ) would save only about 2 launches although, if the mean satellite lifetime was only  $2\frac{1}{2}$  years ( $f = 0.025$ ), an increase of 8 firings would be expected. On the other hand, if the firing rate was halved ( $\ell = 5$  years,  $f = 0.025$ ) e.g. 2 pads each launching at 4 per year, the establishment time would be increased to  $4\frac{1}{4}$  years! Lower launch success probabilities would make these changes even more dramatic.

Clearly, very little can be said here regarding the best manner in which to improve the system in general since it depends upon the state of the various parameters  $\ell$ ,  $p$ ,  $f$ ,  $N$  and the cost of changing any of these quantities. Even so the various curves allow one to determine the time of establishment of a system, given the various parameters for a particular launch scheme and thus to judge whether such a process is feasible.

After establishment, Figs.18 to 22 show, as a function of  $f$ , the percentage of time that there are  $i$  satellites in orbit for systems of (nominally) 6, 12 and 20 satellites. Although the objective is to maintain  $N$  satellites continuously in orbit there is always a finite probability that this will not be achieved unless the firing rate is infinitely fast ( $f = 0$ ). For instance, for a system of 6 satellites and  $p = 0.7$  (Fig.18) if  $f$  is less than 0.02 there is negligible probability that the number of satellites in orbit would drop below 5 and the probability of 6 satellites is greater than 86%. Consequently, for  $f = 0.02$ , it may be concluded that the corresponding firing rate would be quite suitable for maintaining a system of 5 satellites (although still attempting to maintain 6 satellites) and similarly  $f = 0.04$  would maintain 4 satellites and so on. On the other hand, if  $p$  was only 0.5, then  $f = 0.02$  would only be suitable for maintaining 4 satellites and  $f = 0.04$  for 3 satellites. Conversely, for a given  $f$ , one may determine the number of satellites which one should attempt to maintain in orbit so that at least a certain number should be maintained with a given probability.

For example, if  $f = 0.01$  and  $p = 0.7$ , then from Fig.22(a) in order to maintain 18 or more satellites in orbit a possible strategy would be to attempt to retain 20 satellites functioning. The 20 satellites would be maintained with a probability of 72%, at least 19 with 94% and at least 18 with a probability of 98% and negligible probability of these being less than 18.

Finally it is worth mentioning two facts which have not been considered in this Report and which may have a significant effect. In the first place no account has been taken of the time taken to place a satellite in orbit from its time of firing. Secondly, the fact has not been included that a launching may be attempted at any time, rather than at discrete intervals, after the launch pad preparation time has elapsed from its last firing. In order to assess these effects an entirely new theoretical approach is required, analogous to the theory of the maintenance of machines, but so far a satisfactory theory has not been found.

## 6 CONCLUSION

During the establishment phase of a satellite system, if the number of satellite failures is negligible, i.e. the launching interval is very small compared with the satellite mean lifetime ( $\tau/\ell \approx 0$ ), then the probability distribution of establishing the system of  $N$  satellites is approximately Gaussian (for practical purposes) with a mean of  $N/p$  and variance  $N(1-p)/p^2$ . Consequently a fairly rapid estimate of the likely number of launches for a

given probability of establishment may be obtained without evaluating the exact distributions which have been given in this Report. As the launch success probability is decreased there is a drastic increase in the number of required launches due to the mean of the distribution being inversely proportional to  $p$ . The exact nature of this increase is illustrated in Figs. 1 to 7 as a function of the number of satellites required in orbit.

A finite time interval between launches only affects the number of launches for establishment significantly when the ratio  $\tau/\ell$ , and hence  $f = 1 - e^{-\tau/\ell}$ , is increased above a certain region. This is indicated by the steep slope of the curves near  $f = 0$  in Figs. 15 to 17 and is due to the increased probability of satellite failures as the time interval between launches is increased or the mean lifetime is decreased. For a particular satellite system the effect of varying the launch interval, probability of a successful launch or mean lifetime on the number of launches or the establishment time may be obtained from the curves in these figures.

During the maintenance phase, if it is assumed that any failed satellites may be replaced immediately ( $f = 0$ ), then the probability distribution of the number of firings over a period  $T$  is also approximately Gaussian with a mean of  $\alpha N/p$  and variance  $\alpha N(2-p)/p^2$  where  $\alpha = T/\ell$ . Using this distribution simple calculations may be made to estimate the required firing rate after establishment. Low values of  $p$  will, of course, have a marked effect on the launching rate. This is illustrated in Fig. 12 where the number of launches for maintaining a system of  $N$  satellites ( $N < 30$ ) with a given level of probability and  $\alpha = 4$  is shown for various values of launch success probability.

Using a launch strategy which attempts to maintain a specific number of satellites in orbit so that launches are only made when the number of satellites in orbit falls below this number, gives rise to the probability distribution of the percentage of time that there are more than a certain number of satellites in orbit. Consequently, by attempting to maintain a number of satellites in orbit which is in excess of the number required for communication purposes if there were no failures, the probability of breaks in communication occurring in the actual system may be decreased. Estimates of the surplus number of satellites which must be held in orbit so that a system may function as a communication system with little loss in transmission time have been discussed in section 5. One or two excess satellites appear to be sufficient, for satellite systems requiring less than 20 satellites provided the launch success probability is greater than 70% and  $f$  less than 0.01.

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# Appendix A

## MOMENTS AND LIMITING DISTRIBUTIONS

### A.1 Establishment phase

The moment generating function  $M(s)$  of the establishment distribution, equation (5), is given by

$$\begin{aligned}
 M(s) &= \sum_{n=0}^{\infty} p_N^e(n;p) e^{sn} \\
 &= \sum_{n=N-1}^{\infty} \binom{n-1}{N-1} p^N (1-p)^{n-N} e^{sn} \\
 &= \left[ \frac{pe^s}{1-e^s(1-p)} \right]^N \quad (A1)
 \end{aligned}$$

$$= \sum_{r=0}^{\infty} \mu_r' \frac{s^r}{r!} \quad (A2)$$

where  $\mu_r'$  is the  $r$ th moment about the origin of the distribution  $P_N^e(n,p)$ . The moments about the mean,  $\mu_r$ , are obtained from  $\mu_r'$  by means of the relationship<sup>5</sup>

$$\mu_r = \sum_{j=0}^r \binom{r}{j} \mu_{r-j}' (-\mu_1')^j \quad (A3)$$

or

$$\mu_r' = \sum_{j=0}^r \binom{r}{j} \mu_{r-j} \mu_1'^j \quad (A4)$$

In particular,

$$\mu_2 = \mu_2' - \mu_1'^2 \quad (A5)$$

and

$$\mu_3 = \mu_3' - 3\mu_1' \mu_2' + 2\mu_1'^3 \quad (A6)$$

It also follows by differentiation of equation (A2) that

$$\mu_r' = \left[ \frac{d^r}{ds^r} M(s) \right]_{s=0}. \quad (A7)$$

Thus, from equation (A1),

$$M(s) = \left[ \frac{p}{e^{-s} - q} \right]^N, \quad M(0) = 1$$

and therefore

$$\frac{M'(s)}{M(s)} = \frac{Ne^{-s}}{e^{-s} - q} = \frac{N}{1 - qe^s}. \quad (A8)$$

Hence, the mean is

$$\mu_1' = M'(0) = \frac{N}{p}. \quad (A9)$$

Differentiating (A8),

$$(1 - qe^s) M''(s) - qM'(s) = NM'(s)$$

i.e.

$$(1 - qe^s) M''(s) = (N+q) M'(s) \quad (A10)$$

thus gives the second order moment about the origin as

$$\mu_2' = M''(0) = \frac{N(N+1-p)}{p^2}$$

and hence the variance of the distribution is

$$\sigma^2 = \mu_2 = \mu_2' - \mu_1'^2 = \frac{N(N+1-p)}{p^2} - \frac{N^2}{p^2}$$

i.e.

$$\sigma^2 = \frac{N(1-p)}{p^2}. \quad (A11)$$



Knowing the mean and variance, the moment generating function of the distribution in standard measure  $M_n(s)$  may be obtained by multiplying  $M(s)$  by  $e^{-\frac{sp}{\sigma}}$  after replacing  $s$  by  $s/\sigma$ . Thus, using equations (A9) and (A11)

$$\begin{aligned}
 M_n(s) &= e^{-\frac{sN}{\sqrt{Nq}}} \left[ \frac{p e^{\frac{sp}{\sqrt{Nq}}}}{1 - q e^{\frac{sp}{\sqrt{Nq}}}} \right]^N \\
 &= e^{-\frac{sN}{(Nq)^{\frac{1}{2}}}} \left[ 1 + \frac{1}{p} \left( e^{-\frac{sp}{(Nq)^{\frac{1}{2}}}} - 1 \right) \right]^{-N}
 \end{aligned}$$

and the cumulative function

$$\begin{aligned}
 K_n(s) &= \log M_n(s) \\
 &= -\frac{sN}{(Nq)^{\frac{1}{2}}} - N \log \left\{ 1 + \frac{1}{p} \left[ -\frac{sp}{(Nq)^{\frac{1}{2}}} + \frac{s^2 p^2}{2 N q} + O\left(\frac{s^3}{N^{3/2}}\right) \right] \right\} \\
 &= -N \left[ \frac{s^2 p}{2 N q} - \frac{s^2}{2 N q} + O\left(\frac{s^3}{N^{3/2}}\right) \right] \\
 &= \frac{1}{2} s^2 + O\left(\frac{s^3}{N^{1/2}}\right) .
 \end{aligned}$$

Thus for any finite  $s$ , the cumulative function tends to  $\frac{s^2}{2}$  as  $N \rightarrow \infty$ , i.e. the distribution becomes Gaussian for  $N$  large since all the cumulants  $K_r$  for  $r > 2$  tend to zero.

## A.2 Maintenance phase

The probability of maintaining the system with exactly  $n$  launchings (in the time  $T = \alpha \ell$ ) is

$$p_N^m(n;p) = P_N^m(n;p) - P_N^m(n-1;p)$$

and thus with the aid of equation (19), which is more convenient than equation (18) since only the summations depend upon the variable  $n$ ,

$$p_N^m(n;p) = \sum_{k=0}^n \binom{n-1}{k-1} p^k (1-p)^{n-k} \frac{(aN)^k}{k!} e^{-aN} \quad (A12)$$

Consequently, the corresponding moment generating function is

$$\begin{aligned} M(s) &= \sum_{n=0}^{\infty} p_N^m(n;p) e^{sn} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n-1}{k-1} p^k (1-p)^{n-k} \frac{(aN)^k}{k!} e^{-aN} e^{sn} \\ &= \sum_{k=0}^{\infty} \frac{(aN)^k}{k!} e^{-aN} \left( \frac{p}{1-p} \right)^k \left[ \frac{(1-p)e^s}{1 - (1-p)e^s} \right]^k \\ &= e^{-aN} \exp \left\{ \frac{aNpe^s}{1 - (1-p)e^s} \right\} \end{aligned}$$

i.e.

$$M(s) = \exp \left\{ - \frac{aN(1-e^s)}{1 - (1-p)e^s} \right\} \quad (A13)$$

Thus

$$(1 - qe^s) \log M(s) = -aN(1-e^s)$$

and differentiating,

$$(1 - qe^s) \frac{M'(s)}{M(s)} - qe^s \log M(s) = aNe^s \quad (A14)$$

On putting  $s = 0$  in equation (A14) and noting that  $M(0) = 1$ , then the mean of the distribution is

$$\mu_1' = M'(0) = \frac{\alpha N}{p} . \quad (A15)$$

Rewriting equation (A14) in the form

$$(e^{-s} - q) M'(s) - q M(s) \log M(s) = \alpha N M(s)$$

and differentiating once again gives

$$(e^{-s} - q) M''(s) - e^{-s} M'(s) - q M'(s) \log M(s) - q M'(s) = \alpha N M'(s) .$$

Consequently, on putting  $s = 0$ , and using equation (A15), we have

$$p M''(0) = \frac{\alpha N}{p} (\alpha N + 1 + q) .$$

Hence

$$\mu_2' = M''(0) = \frac{\alpha N}{p^2} (\alpha N + 2 - p) \quad (A16)$$

and the variance of the distribution is

$$\sigma^2 = \mu_2' - \mu_1'^2 = \frac{\alpha N(2-p)}{p^2} . \quad (A17)$$

As in the previous section, the moment generating function of the distribution in standard measure may now be found using the mean and variance as given by equations (A15) and (A17). Thus we have

$$\begin{aligned} M_n(s) &= \exp\left(-\frac{\mu_1'}{\sigma}\right) M\left(\frac{s}{\sigma}\right) \\ &= \exp\left(-\frac{s \alpha N}{p \sigma}\right) \exp\left\{-\frac{\alpha N (1 - e^{s/\sigma})}{1 - q e^{s/\sigma}}\right\} \end{aligned}$$

and, consequently, the cumulative function is

$$\begin{aligned}
K_n(s) &= \log M_n(s) = -\frac{s \alpha N}{p\sigma} - \alpha N \frac{(1 - e^{s/\sigma})}{1 - qe^{s/\sigma}} \\
&= -\frac{s \alpha N}{p\sigma} + \frac{\alpha N}{p} \left( \frac{s}{\sigma} + \frac{1}{2} \frac{s^2}{\sigma^2} \right) \left( 1 + \frac{1-p}{p} \frac{s}{\sigma} \right) + O\left(\frac{s^3 N}{\sigma^3}\right) \\
&= \frac{\alpha N}{2p^2} (2-p) \frac{s^2}{\sigma^2} + O\left(\frac{s^3 N}{\sigma^3}\right) .
\end{aligned}$$

But  $\sigma^2 = \alpha N(2-p)/p^2$  and thus

$$K_n(s) = \frac{s^2}{2} + O\left(\frac{s^3}{N^{1/2}}\right)$$

and the distribution becomes Gaussian as  $N \rightarrow \infty$ .

It is also to be noted that, since  $\alpha$  in equation (13) only appears in the form  $\alpha N$ , the distribution will tend to become Gaussian as  $\alpha \rightarrow \infty$  also.

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## Appendix B

### STATISTICS OF A MARKOV CHAIN

#### B.1 Basic concepts

A Markov chain is defined in terms of the transition probabilities  $p_{ij}^{(n)}$  between state  $s_i$  at step  $n$  and state  $s_j$  at step  $(n+1)$  where  $p_{ij}^{(n)}$  is independent of  $n$  and the outcomes before step  $n$ , and may therefore be denoted by  $p_{ij}$ . Consequently, the probability of being in state  $s_j$  after  $n$  steps starting in state  $s_i$  is simply

$$p_{ij}^{(n)} = [P^n]_{ij} \quad (B1)$$

where  $P$  is the transition matrix with elements  $p_{ij}$ .

The states of a Markov chain may be divided into transient and ergodic sets. The former sets, once left, are never entered again, while the latter, once entered, are never again left. Furthermore, if a state  $s_i$  is the only element of an ergodic set, then it is called an absorbing state and will have the element  $p_{ii} = 1$  with all other entries of this row  $p_{ij} = 0$ . If the only non-transient states of a chain are absorbing it is called an absorbing chain. On the other hand, a regular Markov chain is such that it is possible to reach any state after  $n$  step regardless of the initial state and thus contains a single ergodic set and no transient sets.

The matrix  $P$  given by equation (33) in the main text is regular since whatever the number of satellites in orbit at one instant there is always a finite probability of getting into any other state at a later time. On the other hand, if one was only interested in the establishment phase, the elements  $p_{Ni}$  could be put equal to zero for  $i \neq N$  and  $p_{NN} = 1$  so that the state  $N$  was absorbing and thus  $P$  would generate an absorbing Markov chain. This is useful when one is not interested in what happens after reaching state  $N$  i.e. we may consider the attainment of state  $N$  as the end of the process.

#### B.2 Absorbing Markov chains

In what follows, one will only quote some of the more useful results required in analysing Markov chains. For a proof of the formulae the reader is referred to Kemeny and Snell<sup>6</sup>.

Let  $M_i$  ( $\text{Var}_i$ ) denote the mean (variance) of a function starting from the state  $i$  and  $n_j$  the total number of times in the transient state  $s_j$ . Also let

$Q$  be the transient part of the transition matrix  $P$  obtained by deleting all those rows and columns containing a unit element. Then, for an absorbing Markov chain, the fundamental matrix  $N$  is defined by

$$N = (I - Q)^{-1} \quad (B2)$$

and

$$\{M_i[n_j]\} = N \quad (B3)$$

Also

$$\{M_i[n_j^2]\} = N(2N_{dg} - I) \quad (B4)$$

where  $N_{dg}$  is the matrix obtained from  $N$  by putting all off diagonal elements equal to zero. Consequently, the variance of  $n_j$  is

$$\{Var_i[n_j]\} = N(2N_{dg} - I) - N_{sq} \quad (B5)$$

where  $N_{sq}$  is the matrix obtained from  $N$  by squaring each element.

Finally, if  $t$  is the number of steps in which the process is in a transient state, then

$$\{M_i[t]\} = N\xi \equiv \tau, \quad (B6)$$

$$\{M_i[t^2]\} = (2N - I)\tau$$

so that

$$\{Var_i[t]\} = (2N - I)\tau - \tau_{sq} \quad (B7)$$

where  $\xi$  denotes a column matrix with unit elements.

For the establishment phase of satellites where the  $N$ th state is considered absorbing, equations (B6) and (B7) thus give the mean and variance of the time required to establish the system.

### B3 Regular Markov chains

If  $P$  is regular then  $P^n$  tends to a constant probability matrix  $A$  as  $n \rightarrow \infty$  with all its rows equal to the same probability vector  $\alpha = (a_0, a_1, \dots, a_N)$  and

$$\alpha P = \alpha \quad (B8)$$

This means that any long range predictions are independent of the initial state. Also  $a_j$  represents the fraction of time that the system can be expected to be in state  $s_j$  over a long period of time.

For regular chains, the so called fundamental matrix is defined by

$$Z = [I - (P - A)]^{-1} \quad (B9)$$

and is the basic quantity used to compute most of the interesting descriptive quantities.

If  $f_j$  denotes the number of steps before entering state  $s_j$  for the first time, and if we denote  $\{M_i[f_j]\}$  by the matrix  $M = \{m_{ij}\}$ ,  $M$  is given by

$$M = (I - Z + E Z_{dg})R \quad (B10)$$

where  $R$  is the diagonal matrix with elements  $r_{ii} = 1/a_{ii} = m_{ii}$  and  $E$  is a square matrix composed of unit entries. Thus  $M$  is a measure of the mean number of steps to the first passage through any state. In particular  $m_{oj}$  is the mean number of launchings to establish  $j$  satellites in orbit in the context of this Report.

The variance of the first passage time may also be obtained since, if  $W = \{M_i[f_j^2]\}$ , then  $W$  is given by

$$W = M(2Z_{dg} R - I) + 2 (ZM - E(ZM)_{dg}) \quad (B11)$$

and consequently

$$\text{Var}_i[f_j] = W - M_{sq} \quad (B12)$$

SYMBOLS

$a_i$	elements of vector $a$
$A$	matrix with all rows equal to the limiting probability vector $a$
$B$	'birth' matrix, see equation (36)
$B_N(n,p)$	binomial distribution, see equation (4)
$B(\alpha, \beta)$	complete Beta function = $B_1(\alpha, \beta)$
$B_p(\alpha, \beta)$	incomplete Beta function
$D$	'death' matrix, see equation (35)
$E$	square matrix composed of unit entries
$f = 1-s$	satellite failure probability during interval $\tau$
$f_j$	number of steps before reaching state $j$ for the first time
$G_r$	Gaussian distribution, see equation (29)
$I$	unit matrix
$I(\alpha, \beta)$	incomplete Gamma function (Pearson tables), see equation (25)
$I_p(\alpha, \beta)$	incomplete Beta function (Pearson tables), see equation (11)
$K(s)$	cumulative generating function
$\ell$	mean lifetime of a satellite
$m$	mean of a distribution
$m_{ij}$	mean number of steps starting in state $i$ before reaching state $j$ for the first time
$M$	matrix with elements $m_{ij}$ defined by equation (B10)
$M_i$	mean of a function starting in state $i$
$M(s)$	moment generating function
$n$	number of launches
$n_j$	number of times in a transient state $j$
$N$	number of satellites in complete system; fundamental matrix of an absorbing Markov chain, see equation (B2)
$p$	probability of a successful launch
$p_{ij}$	elements of transition matrix $P$
$p(t)$	Poisson probability density of satellite lifetime, see equation (1)
$p_N^e(n;p)$	probability density for establishment
$p_N^m(n;p)$	probability density for maintenance
$p(k;\alpha)$	Poisson frequency distribution, see equation (14)
$P$	transition matrix
$P(t)$	see equation (2)
$P_N^e(n;p)$	probability distribution for establishment



$P_N^m(n;p)$	probability distribution for maintenance
$P(c,a)$	see equation (20)
$q$	$= 1-p$
$Q$	transient part of the transition matrix
$R$	diagonal matrix with elements $m_{ii}$
$s$	$= 1-f$ , satellite survival probability during interval $\tau$
$t$	time
	number of launches before system established
$T$	maintenance period
$\text{Var}_i$	variance of a function starting in state $i$
$W$	see equation (B11)
$Z$	fundamental matrix for a regular Markov chain, see equation (B9)
$\alpha$	$= T/\ell$ , see section 4
	or limiting probability vector $\alpha$ , see equation (B8)
$\Gamma$	complete Gamma function
$\Gamma_a$	incomplete Gamma function
$\mu_r$	$r$ th moment about the origin
$\mu_r'$	$r$ th moment about the mean
$\xi$	column matrix with unit entries
$\sigma^2$	variance of a distribution
$\tau$	interval between launches

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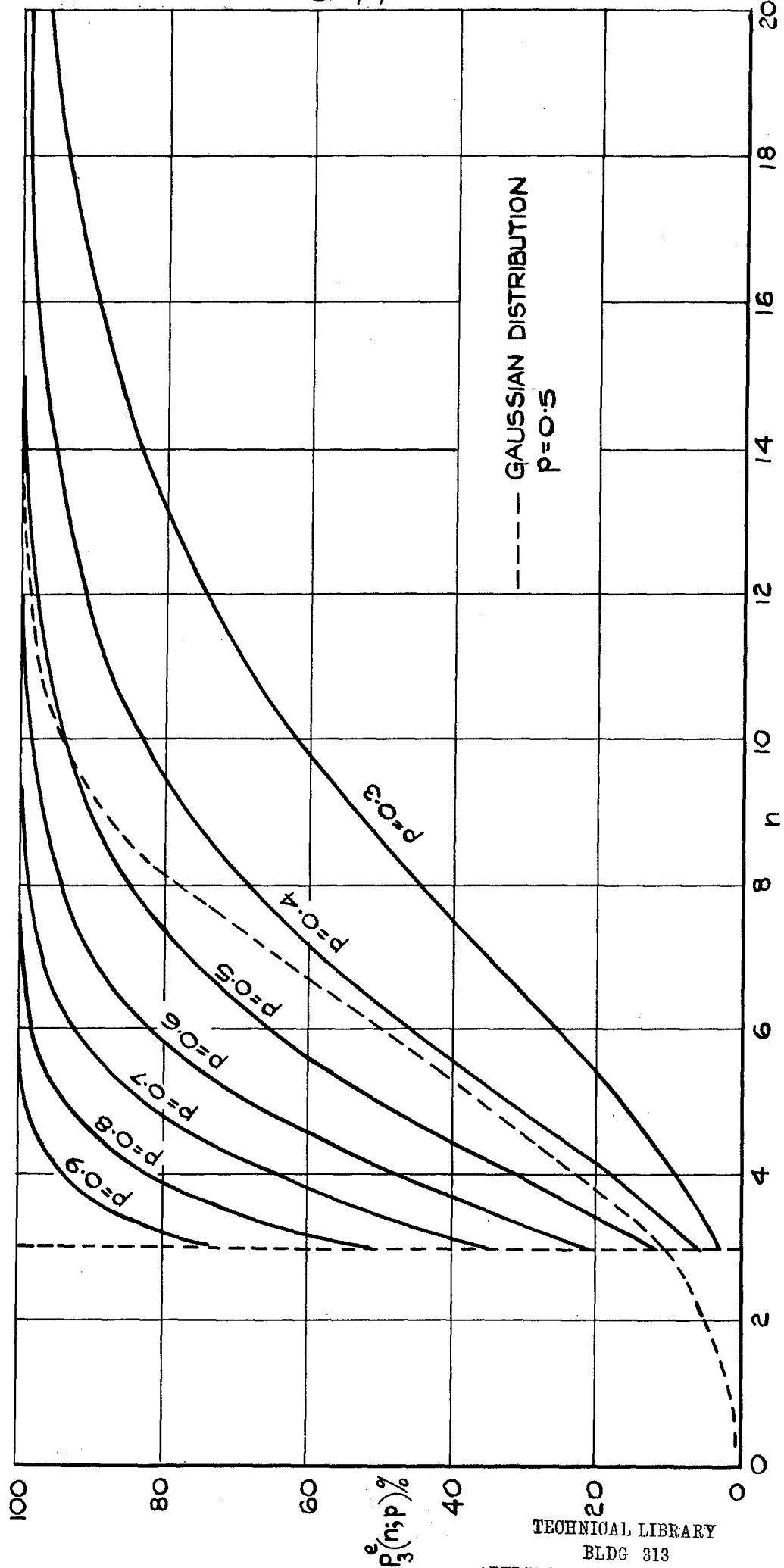
FIG.1 ESTABLISHMENT PROBABILITY FOR HIGH LAUNCH RATES ( $N=3$ )

Fig.2

SPA/P/1669

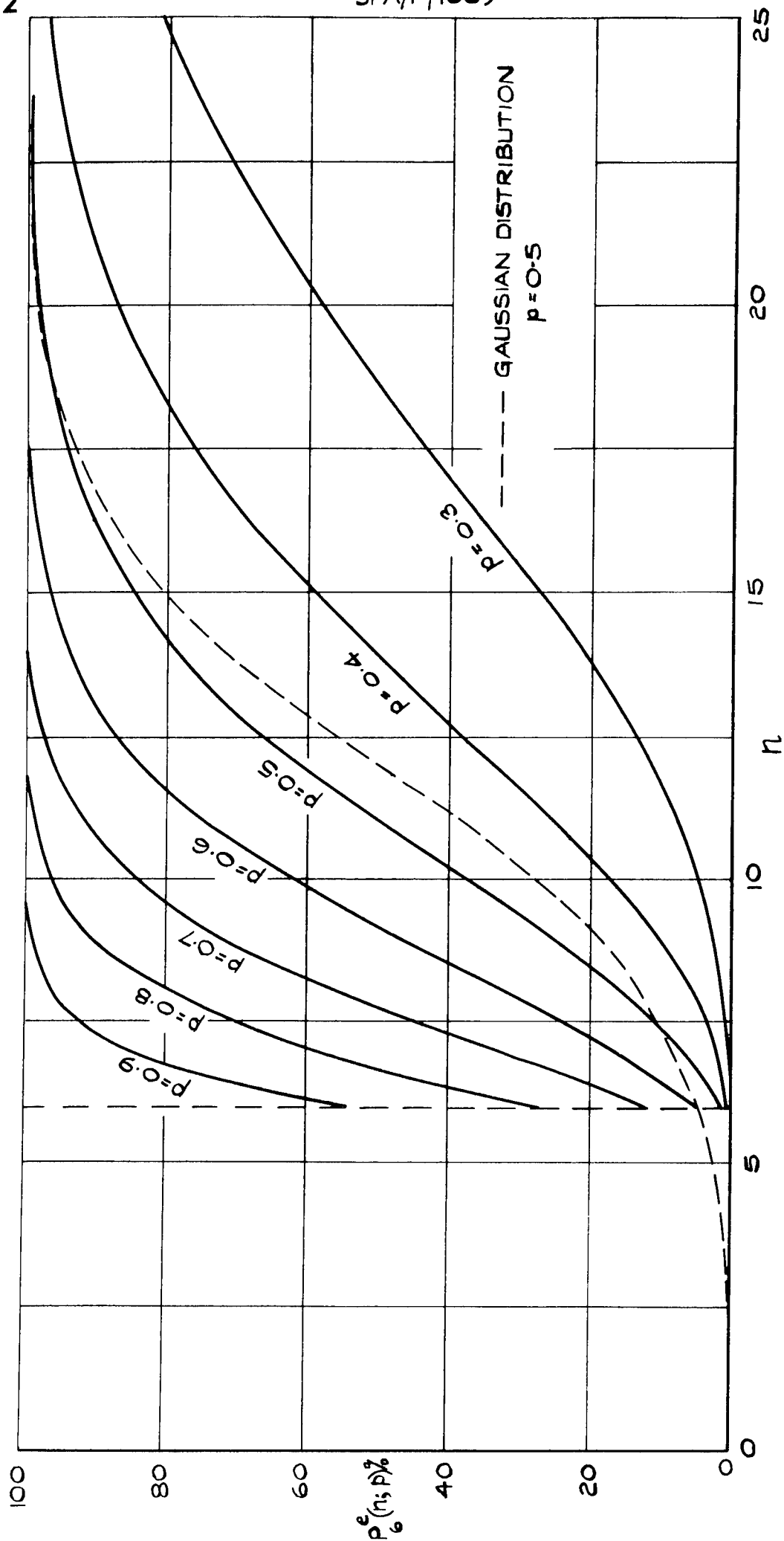


FIG.2 ESTABLISHMENT PROBABILITY FOR HIGH LAUNCH RATES ( $N = 6$ )

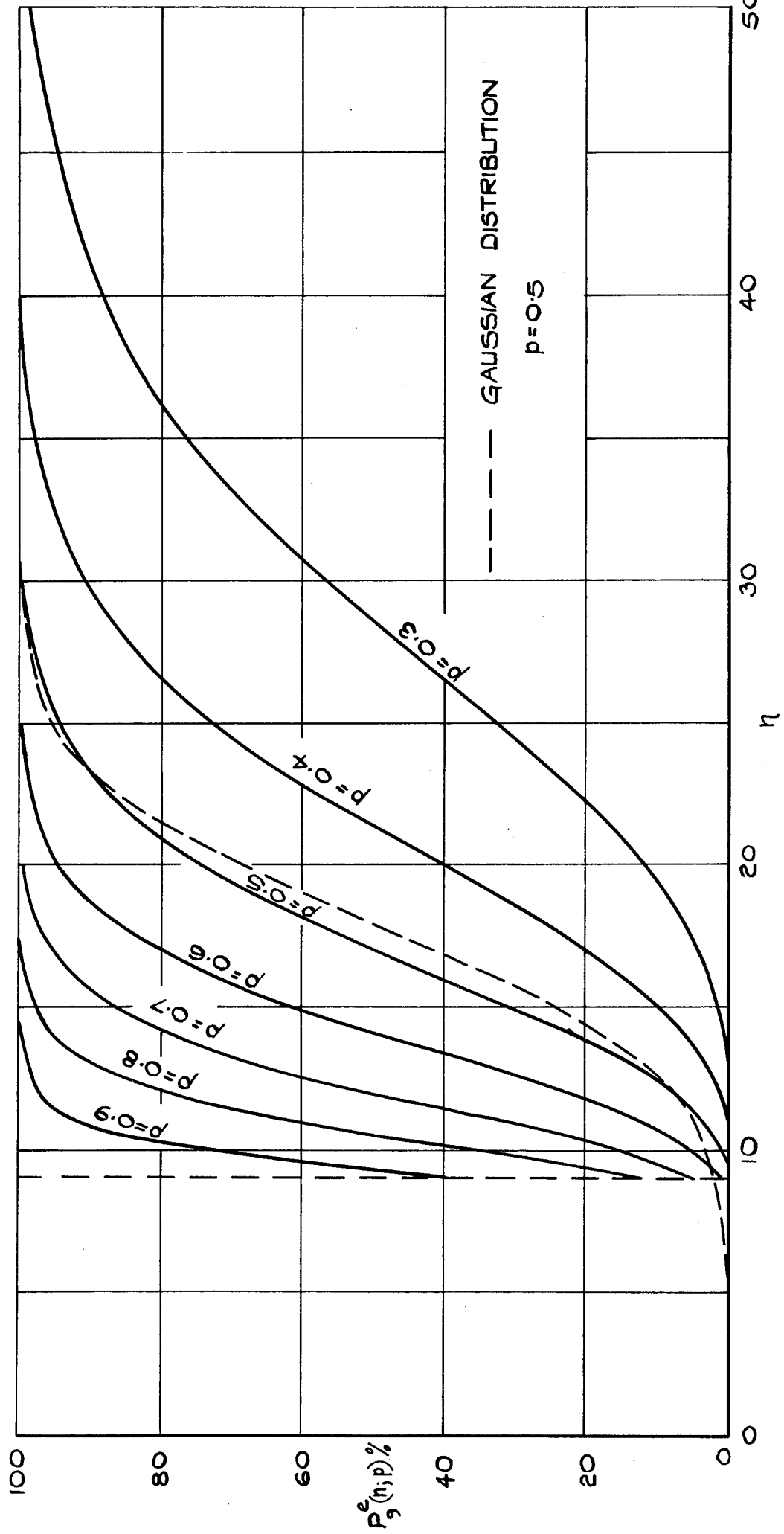
FIG.3 ESTABLISHMENT PROBABILITY FOR HIGH LAUNCH RATES ( $N=9$ )

Fig.4

SPA/P/1671

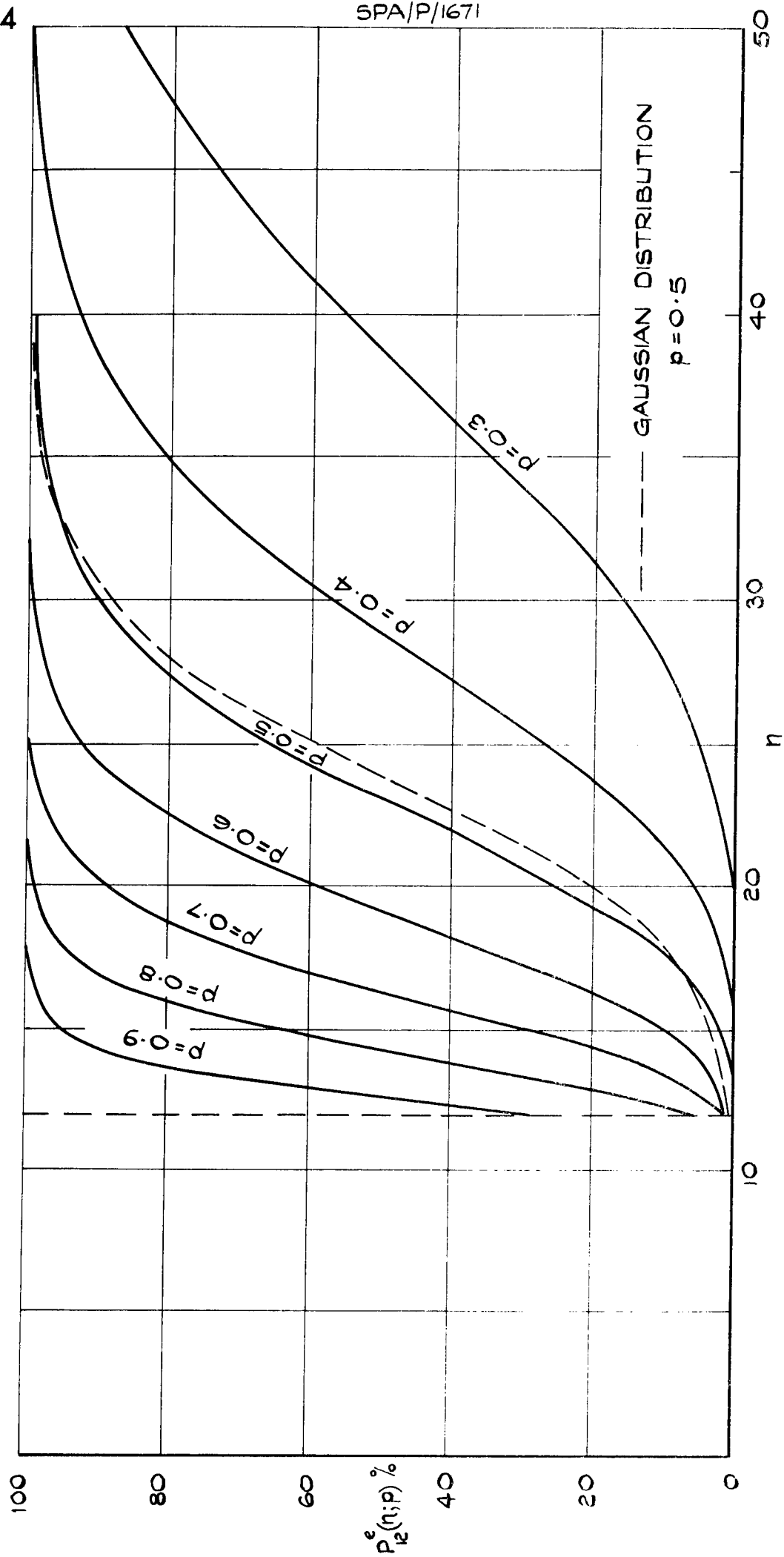


FIG.4 ESTABLISHMENT PROBABILITY FOR HIGH LAUNCH RATES ( $N=12$ )

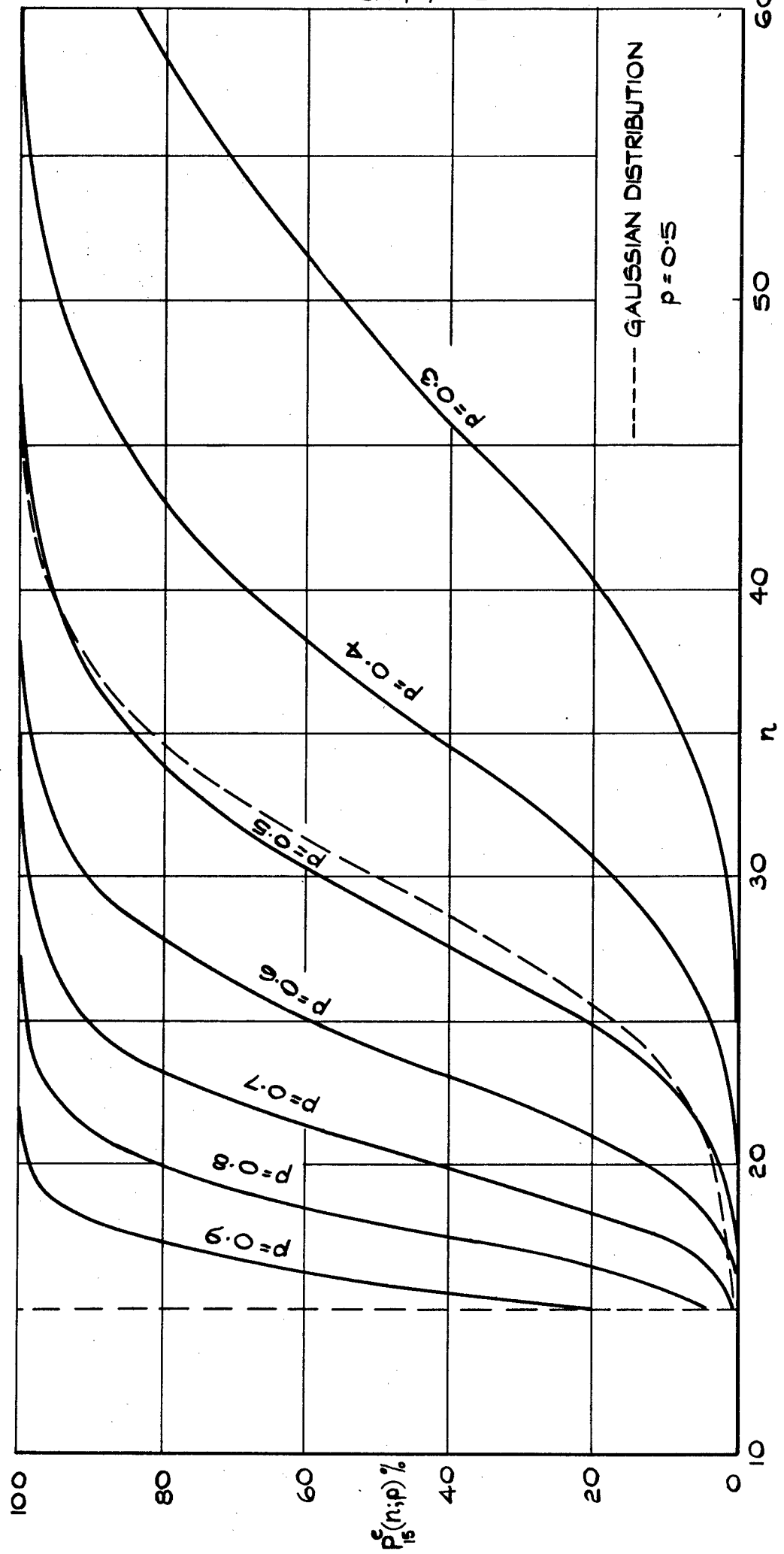
FIG.5 ESTABLISHMENT PROBABILITY FOR HIGH LAUNCH RATES ( $N=15$ )

Fig.6

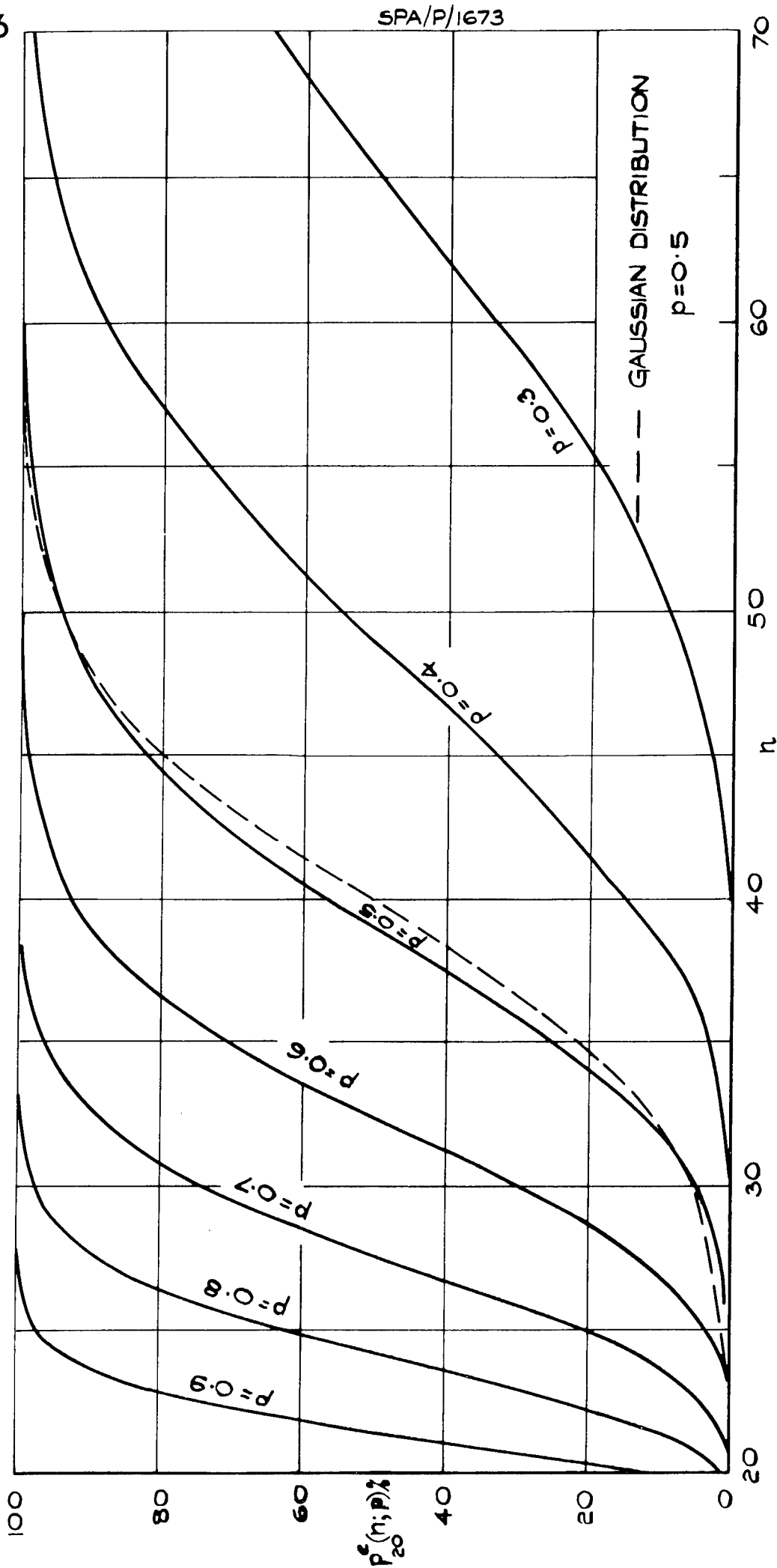


FIG.6 ESTABLISHMENT PROBABILITY FOR HIGH LAUNCH RATES ( $N = 20$ )



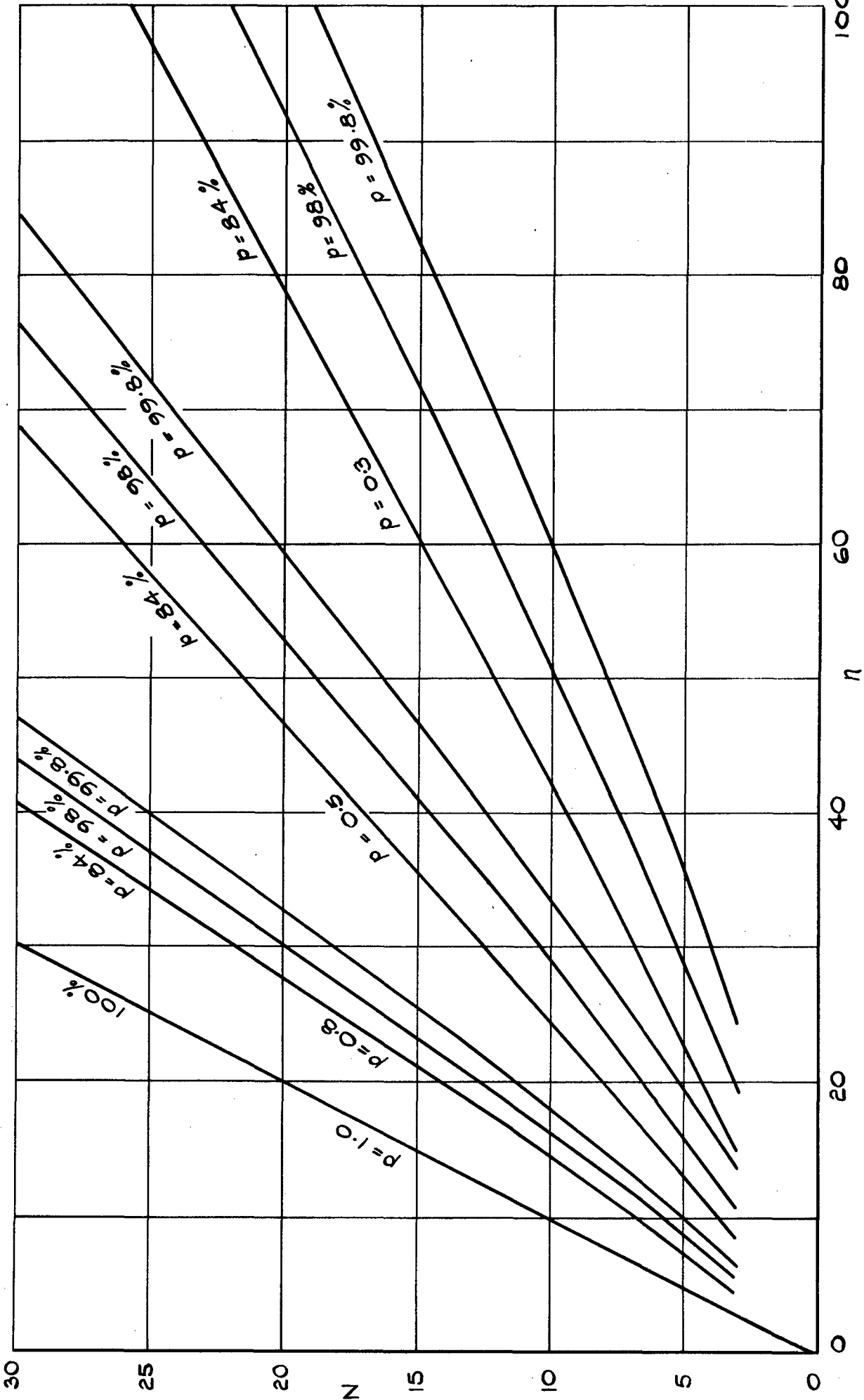


FIG.7 NUMBER OF LAUNCHES FOR THE ESTABLISHMENT OF N SATELLITES (GAUSSIAN DISTRIBUTION)

Fig.8

SPA/P/1675

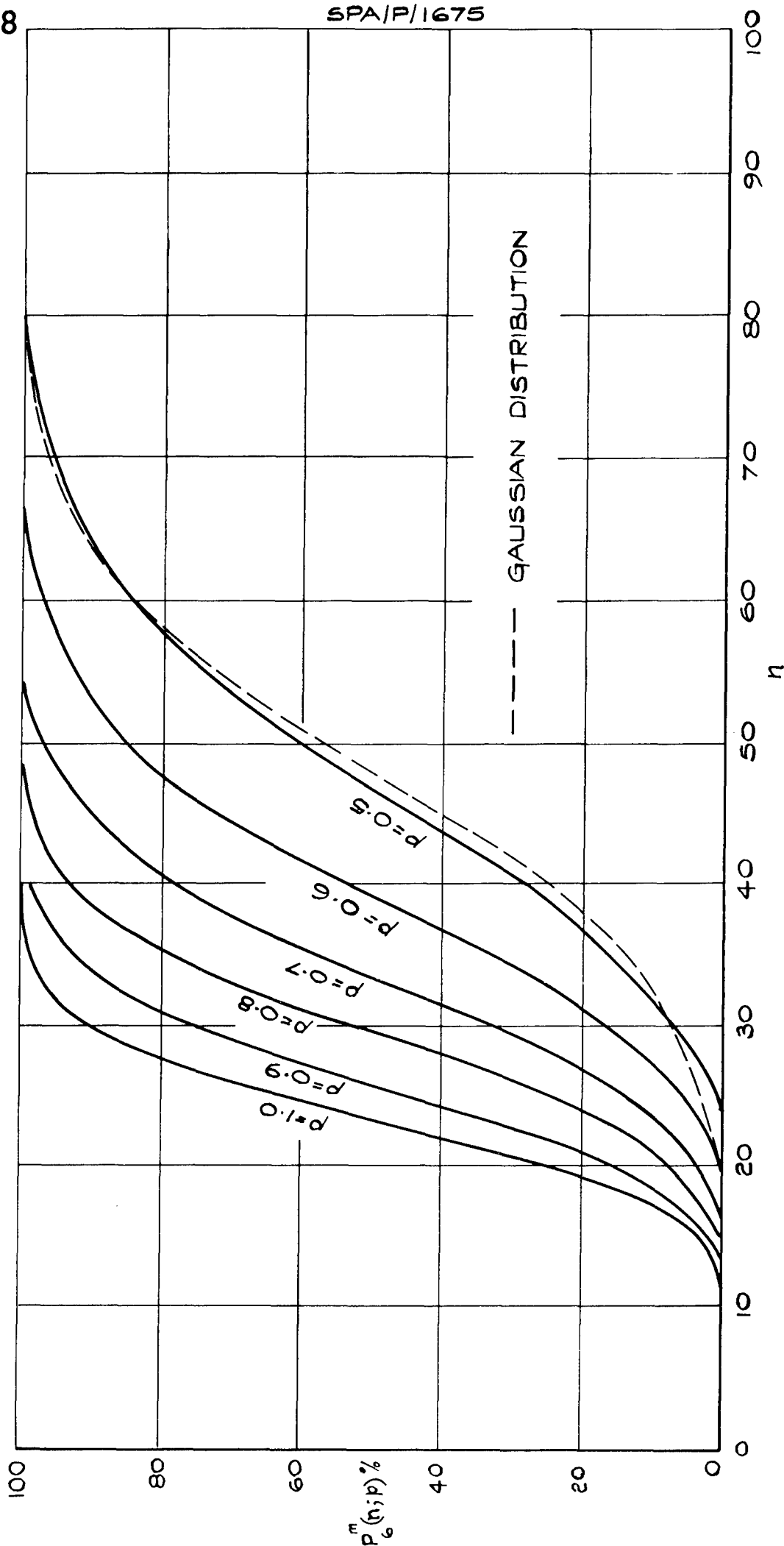


FIG. 8 MAINTENANCE PROBABILITY DISTRIBUTION  
FOR  $N = 6$  AND  $T/l = 4$  ( $\alpha N = 24$ )

Fig.9

$N=9 \quad \bar{n}=36$

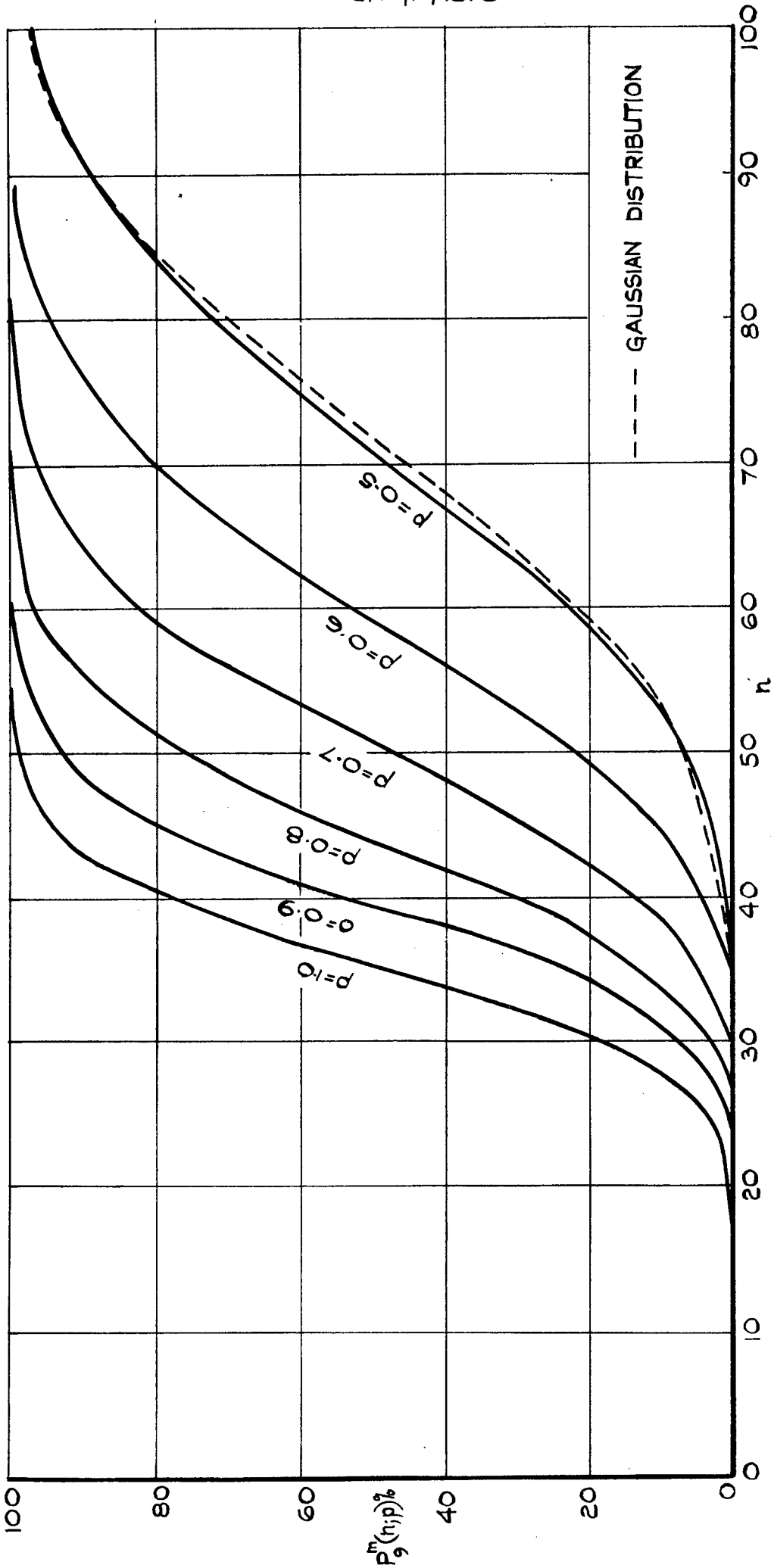


FIG. 9 MAINTENANCE PROBABILITY DISTRIBUTION FOR  $N=9$  AND  $T/\ell=4 (\propto N=36)$

Fig.10

SPA/P/1677

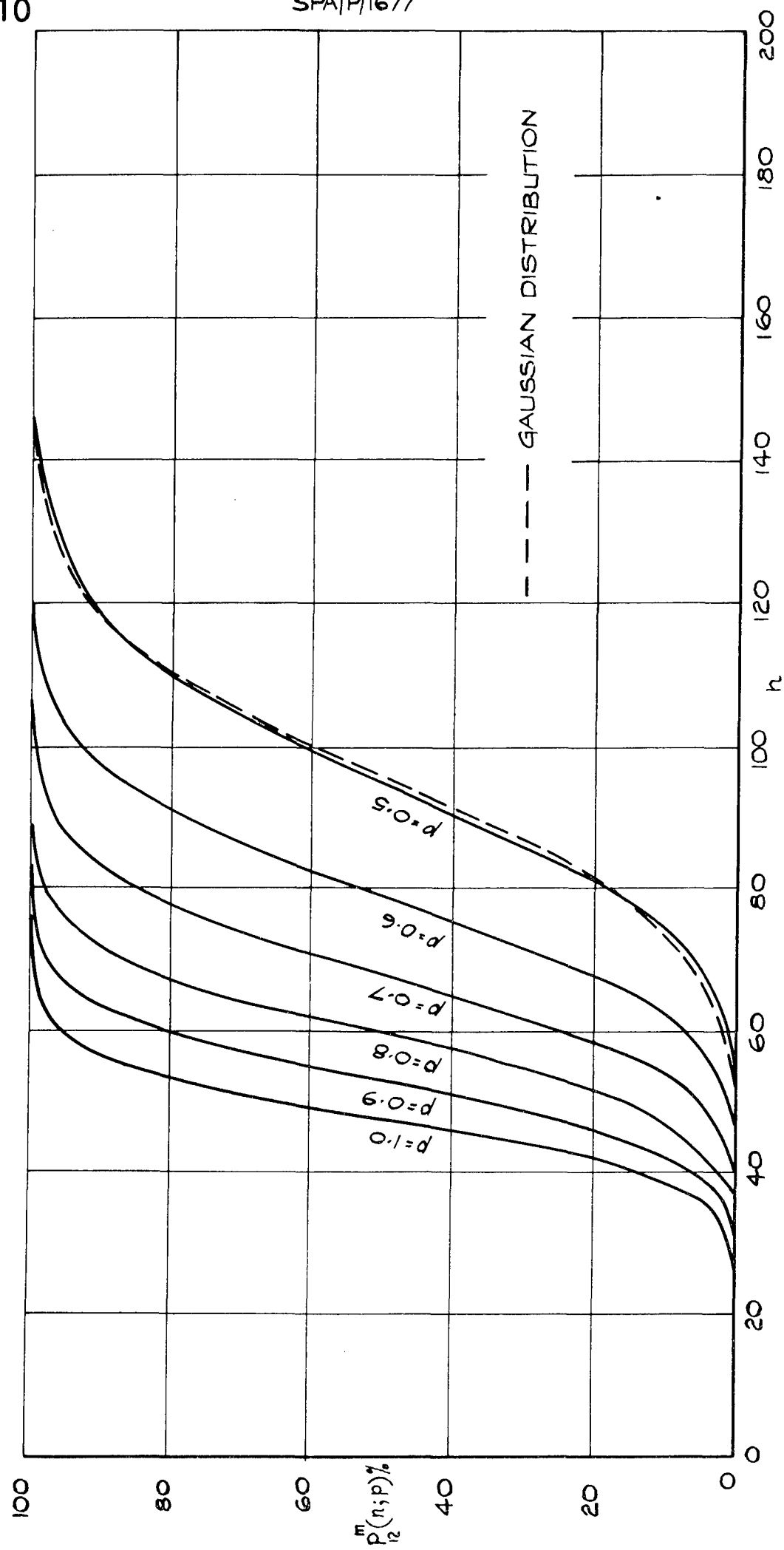


FIG.10 MAINTENANCE PROBABILITY DISTRIBUTION FOR  $N=12$  AND  $T/p=4$  ( $\alpha N=48$ )

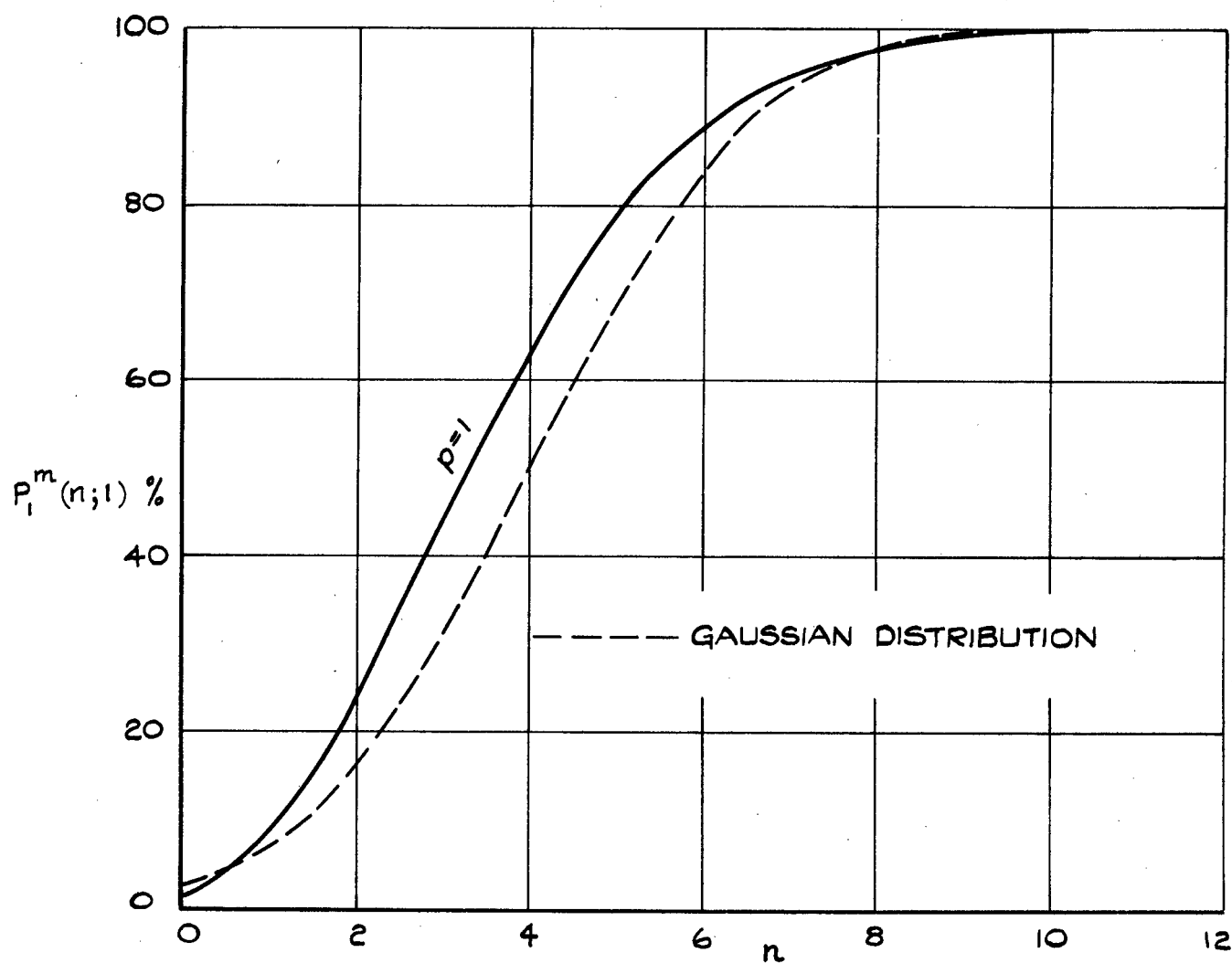


FIG.II MAINTENANCE PROBABILITY DISTRIBUTION  
FOR  $N=1$  AND  $T/l=4$  ( $\propto N=4$ )

Fig.12

SPA/P/1679

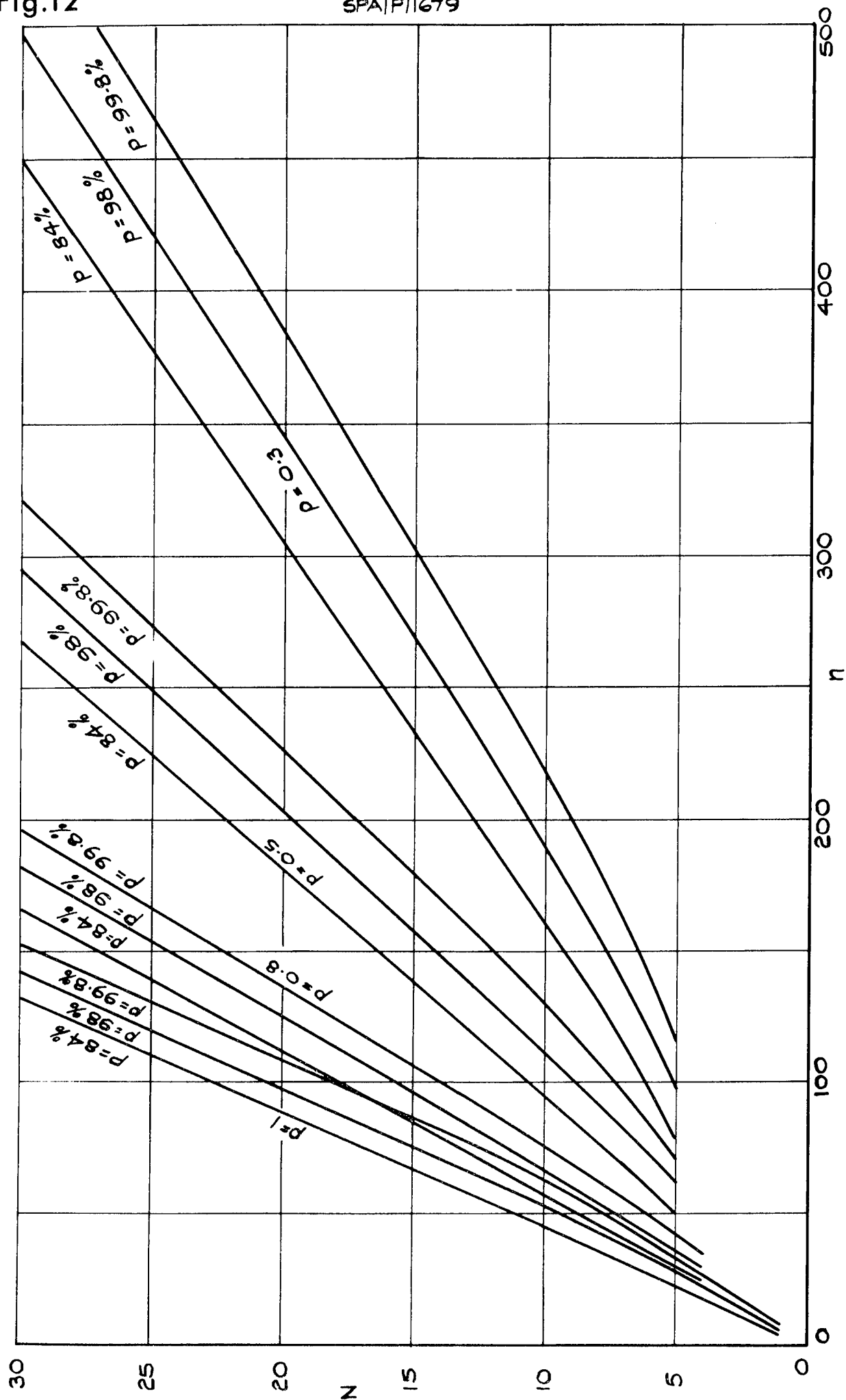


FIG. 12 NUMBER OF LAUNCHES FOR THE MAINTENANCE OF  
N SATELLITES WITH  $T/l = 4$  (GAUSSIAN DISTRIBUTION)

Fig.13

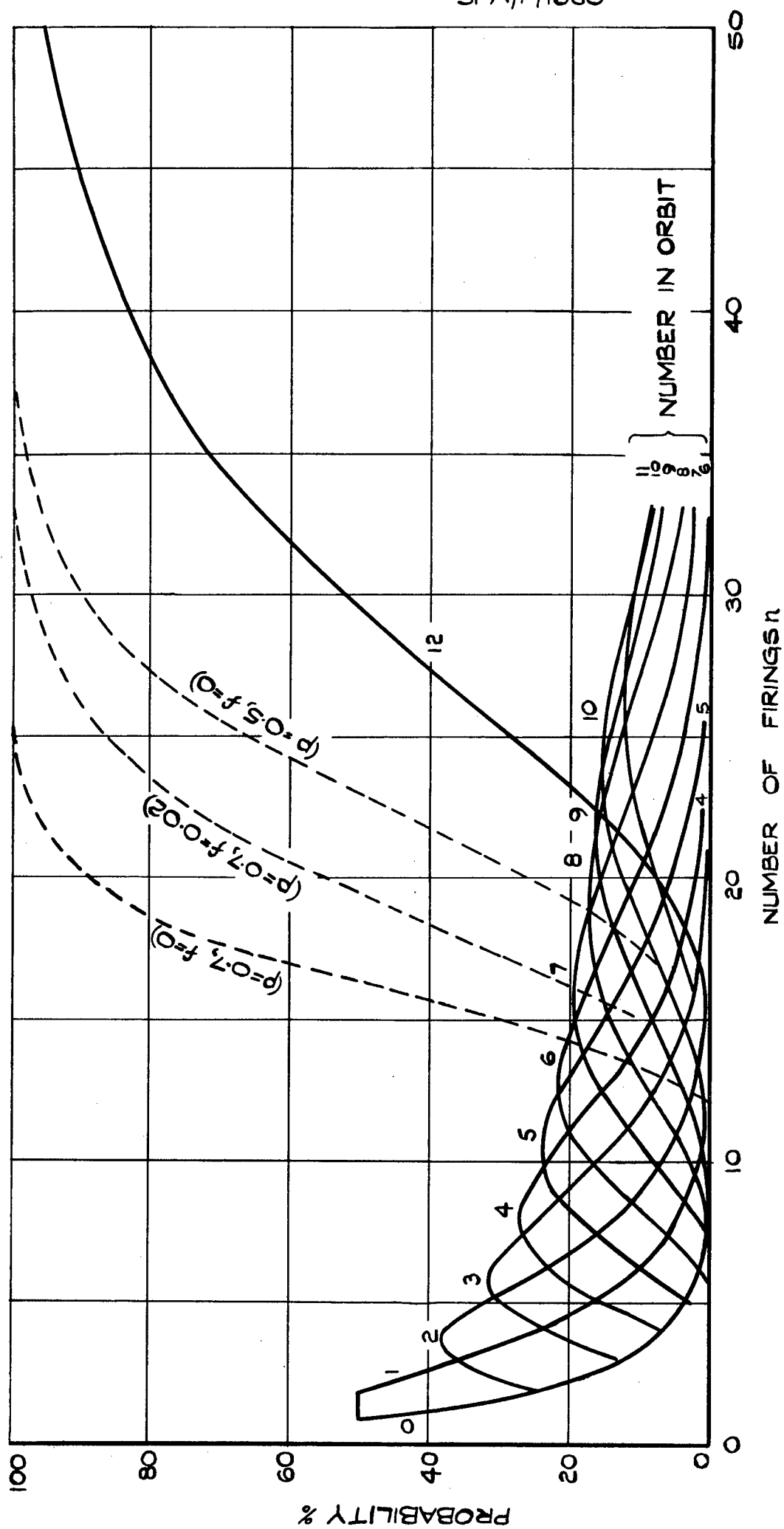


FIG.13 PROBABILITY OF NUMBER OF SATELLITES IN ORBIT AFTER  $n$  LAUNCHINGS  
 $(N=12, p=0.5, f=0.02)$

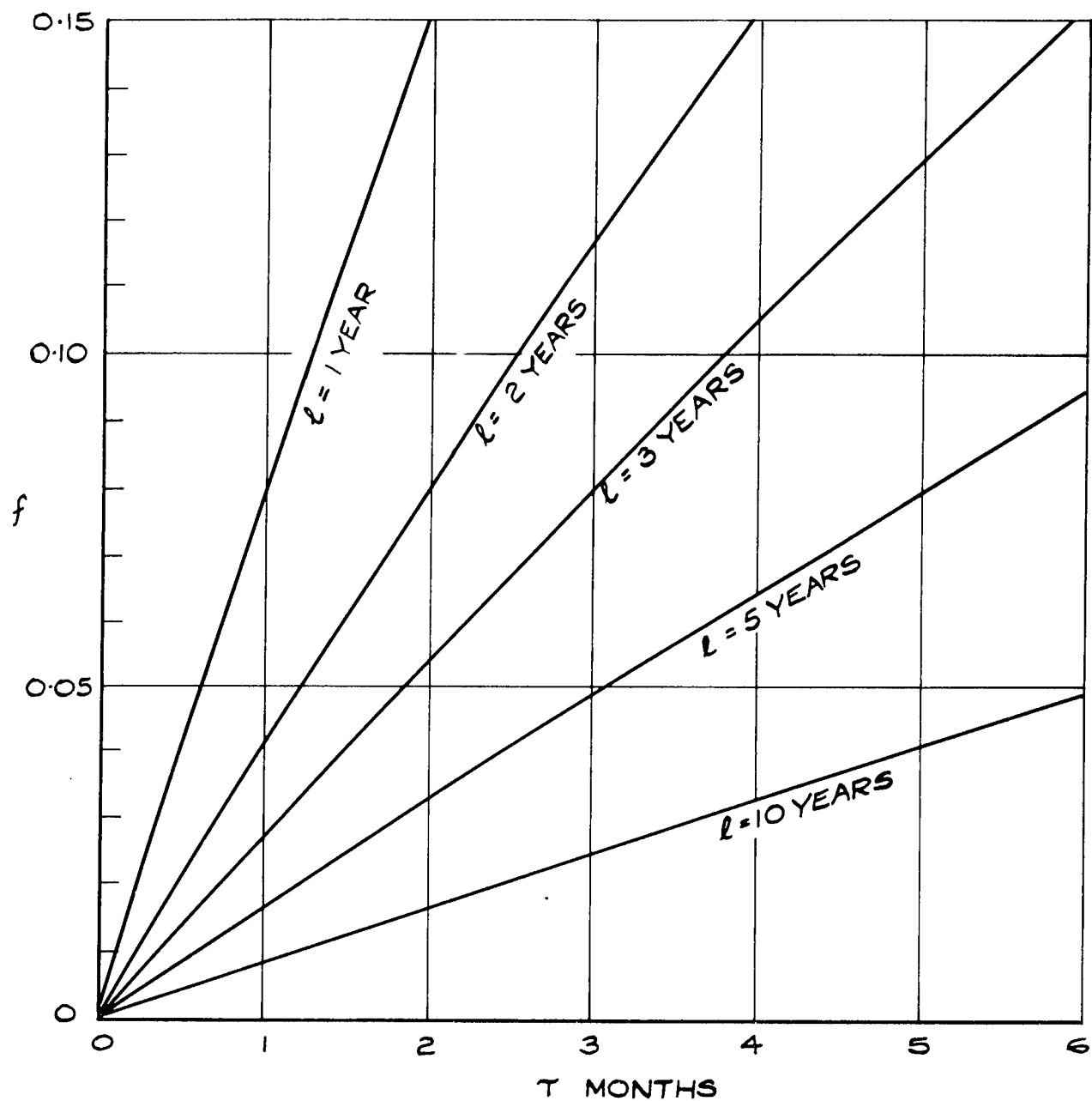


FIG.14 PROBABILITY OF SATELLITE FAILURE ( $f$ ) AS A FUNCTION OF LAUNCH PERIOD ( $\tau$ ) AND SATELLITE MEAN LIFETIME ( $\ell$ )



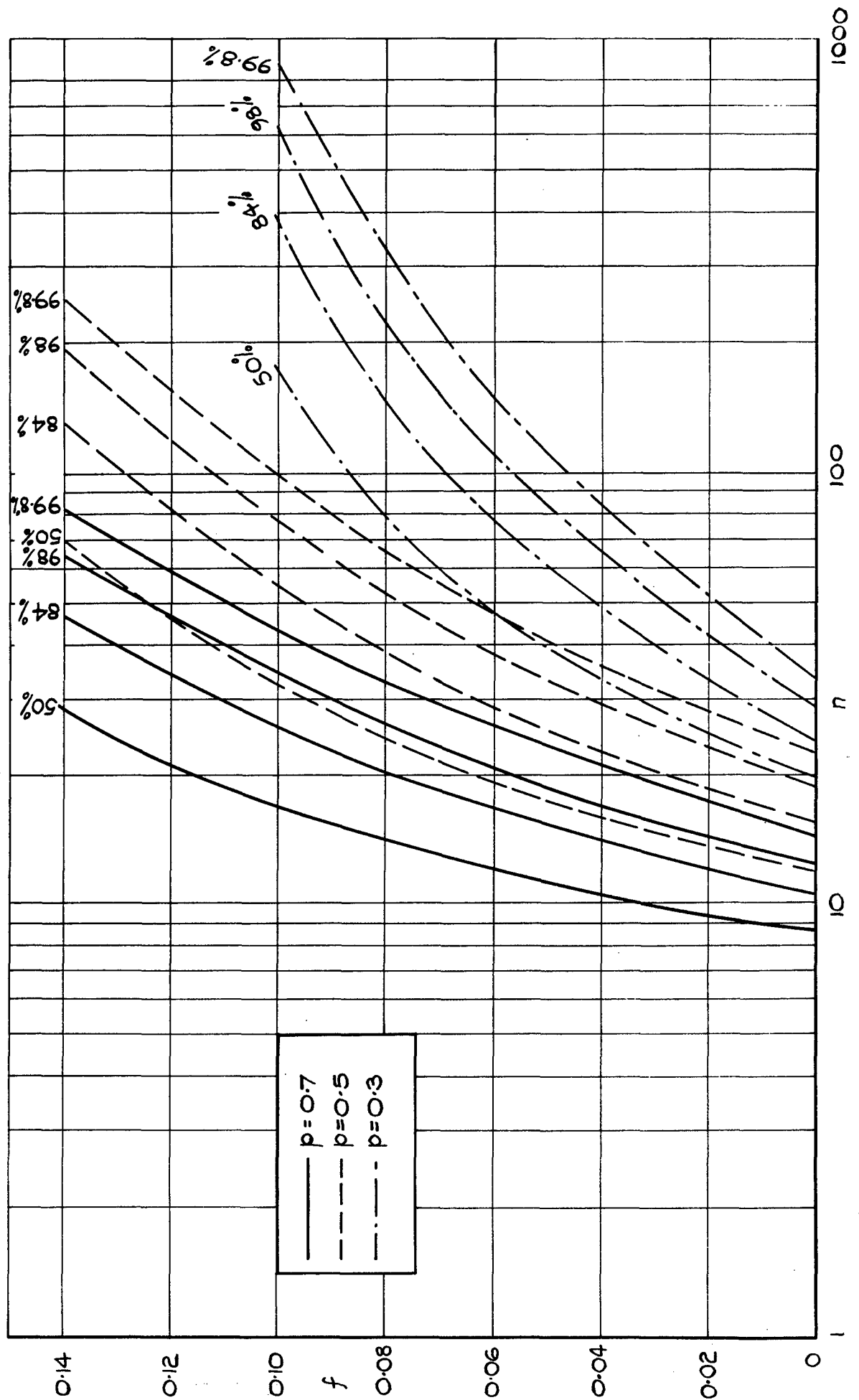


FIG. 15 NUMBER OF LAUNCHES FOR THE ESTABLISHMENT OF 6 SATELLITES AS A FUNCTION OF  $f$

Fig.16

SPA/P/1683

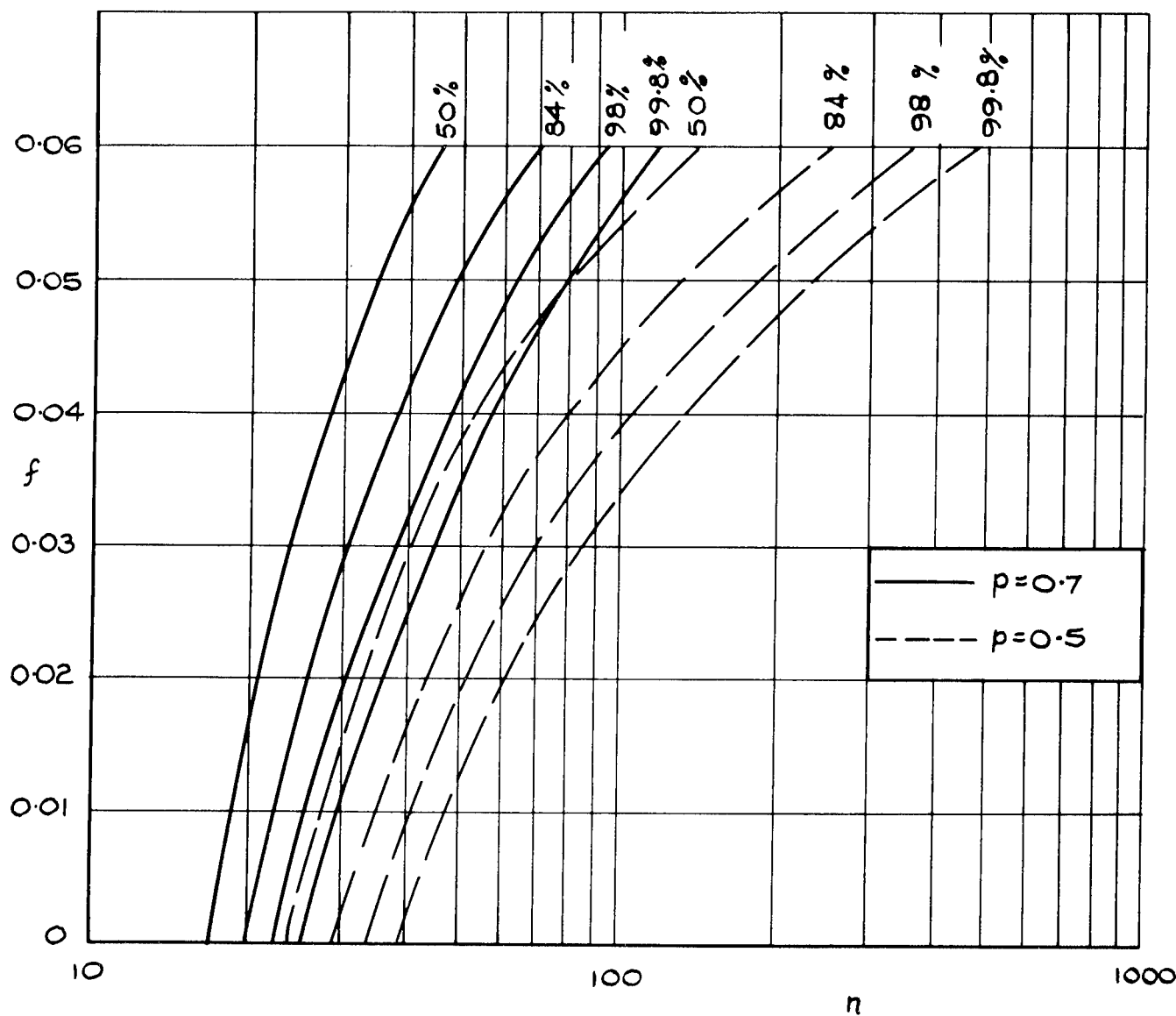


FIG.16 NUMBER OF LAUNCHES FOR THE ESTABLISHMENT OF 12 SATELLITES AS A FUNCTION OF  $f$

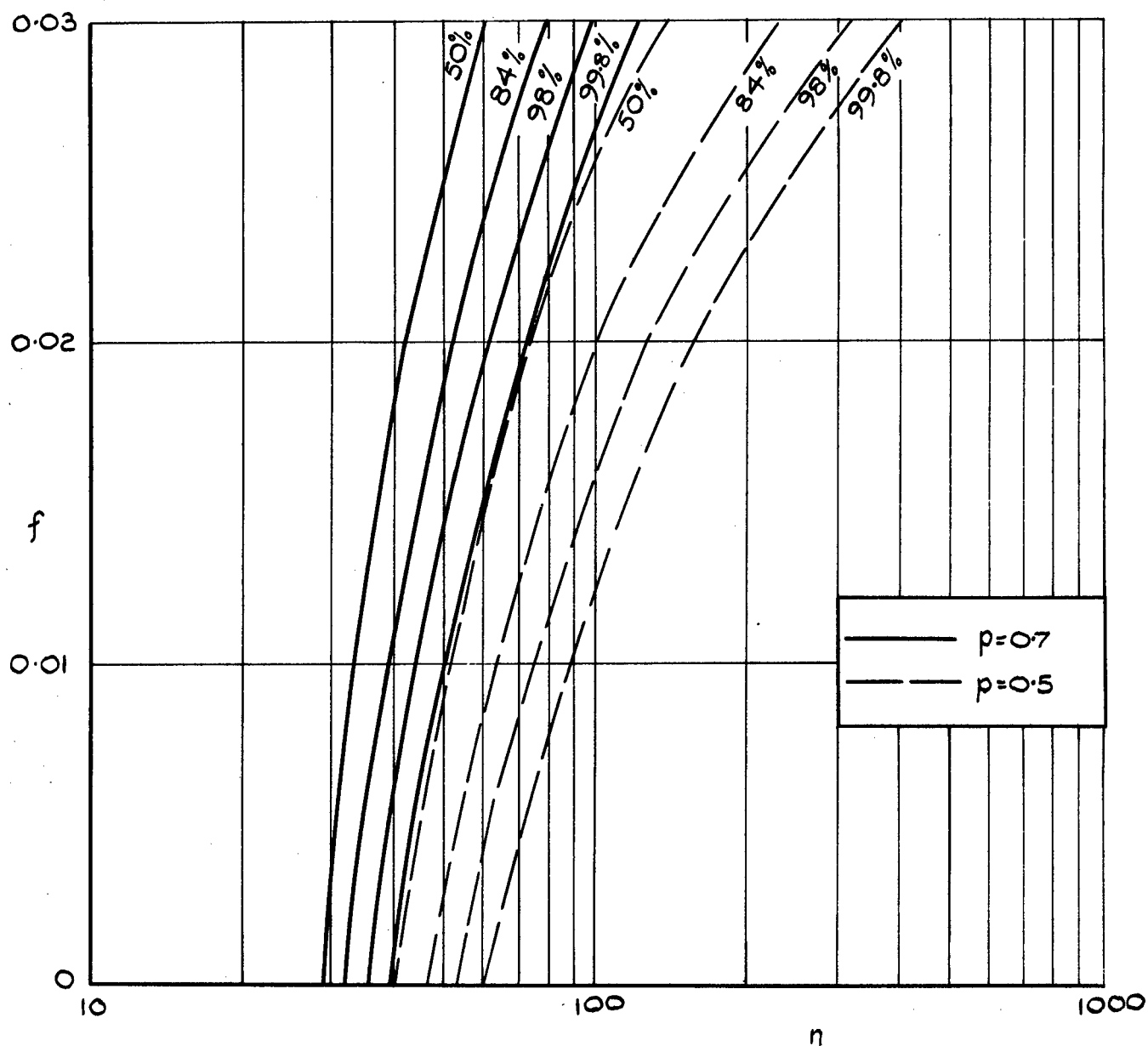


FIG.17 NUMBER OF LAUNCHES FOR THE ESTABLISHMENT OF 20 SATELLITES AS A FUNCTION OF  $f$

Fig.18

SPA/P/1685

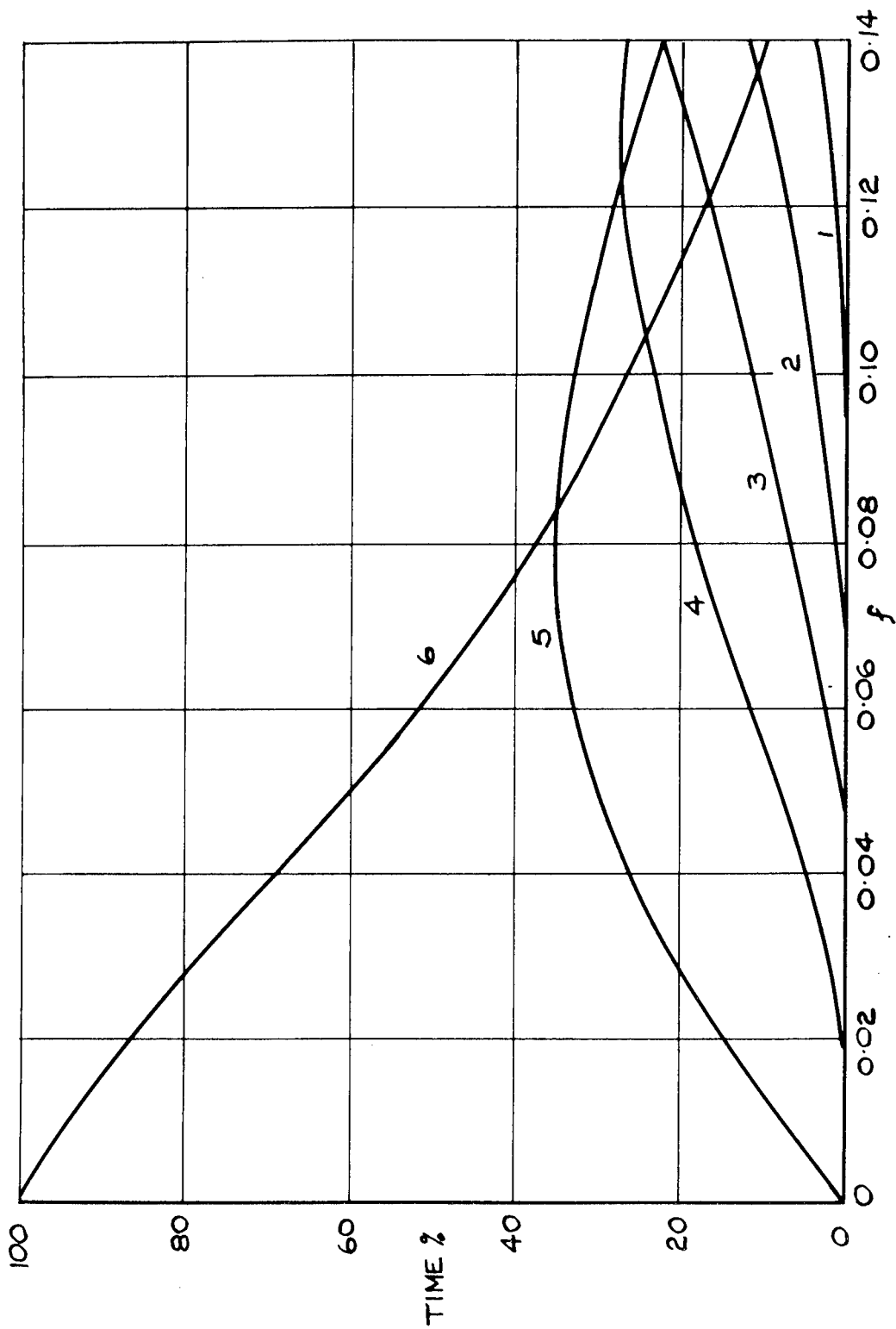


FIG.18 STATE DURATIONS AS A FUNCTION OF  $f$  FOR  $N=6$  AND  $p=0.7$

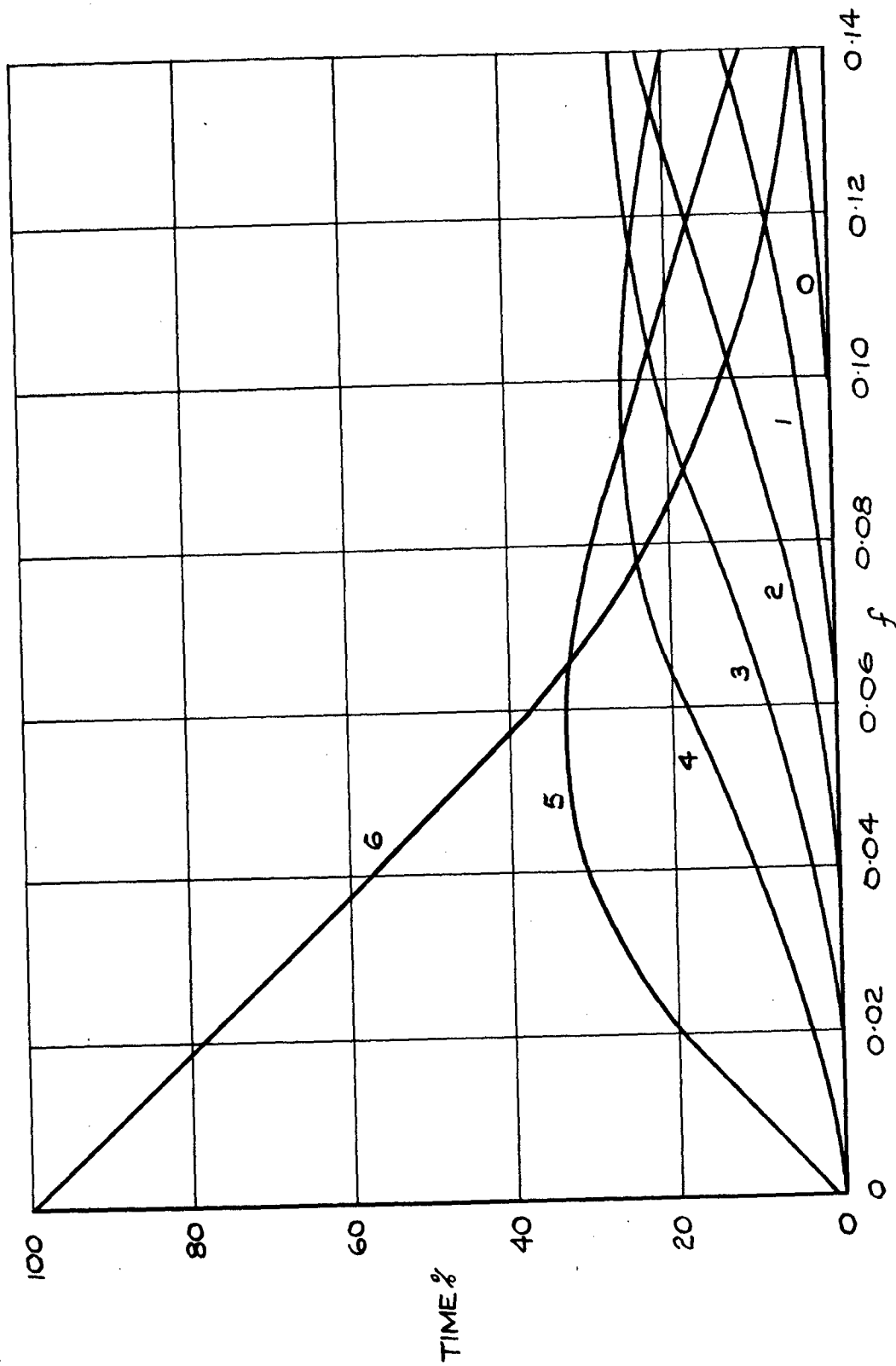


FIG.19 STATE DURATIONS AS A FUNCTION OF  $f$  FOR  $N=6$  AND  $p=0.5$

Fig.20

SPA/P/1687

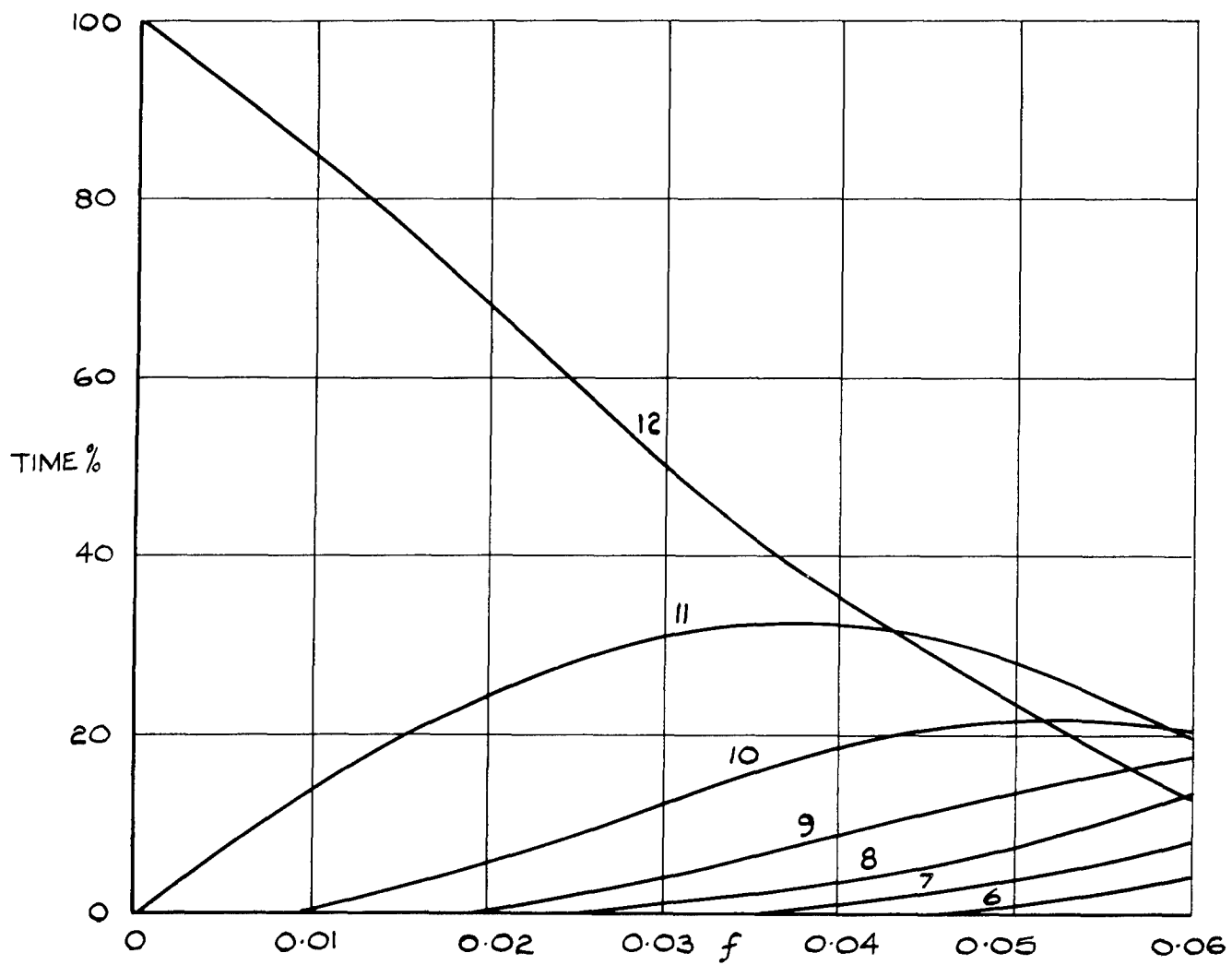


FIG. 20 STATE DURATIONS AS A FUNCTION OF  $f$  FOR  $N=12$  AND  $p=0.7$

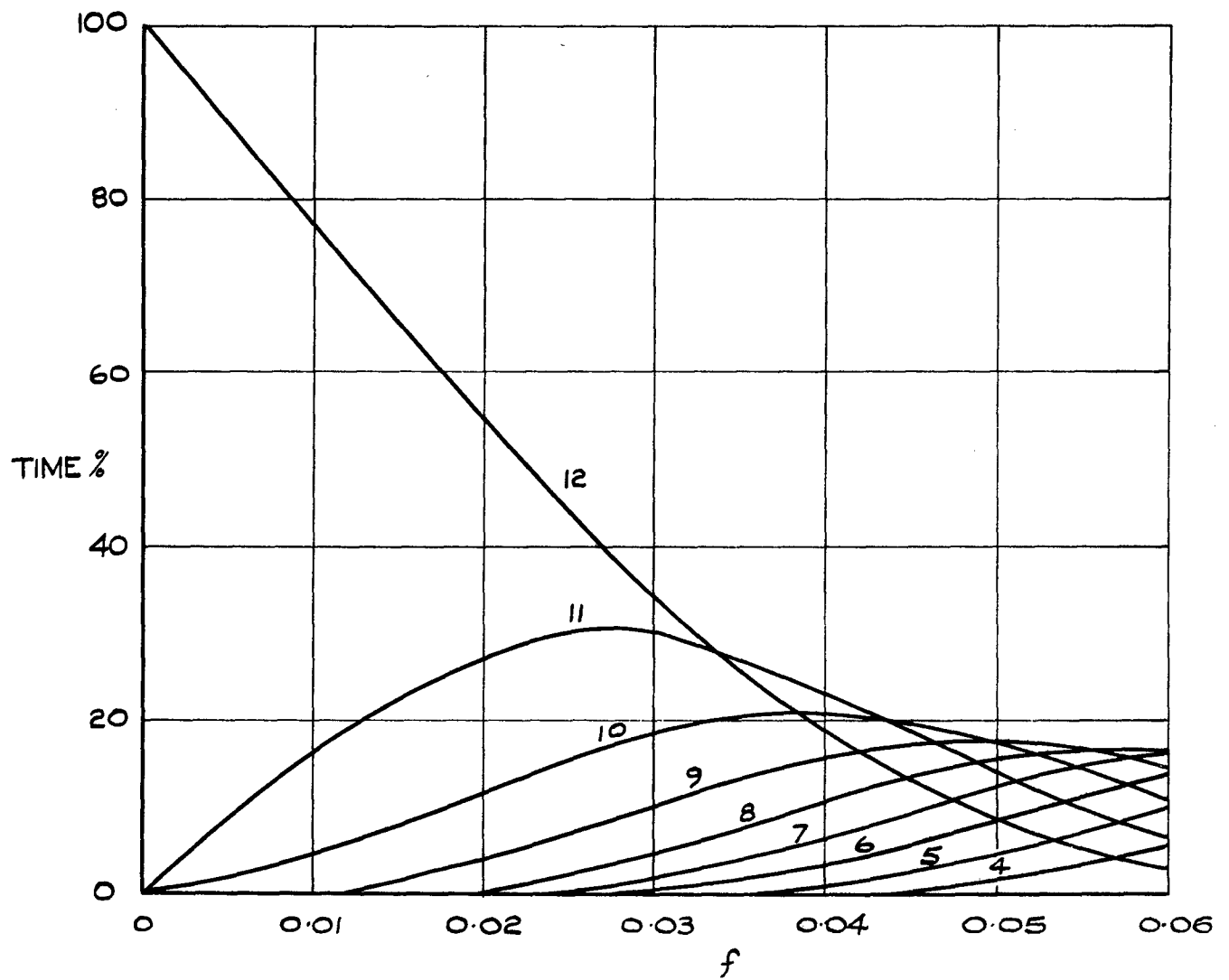


FIG. 21. STATE DURATIONS AS A FUNCTION OF  $f$  FOR  $N=12$  AND  $p=0.5$

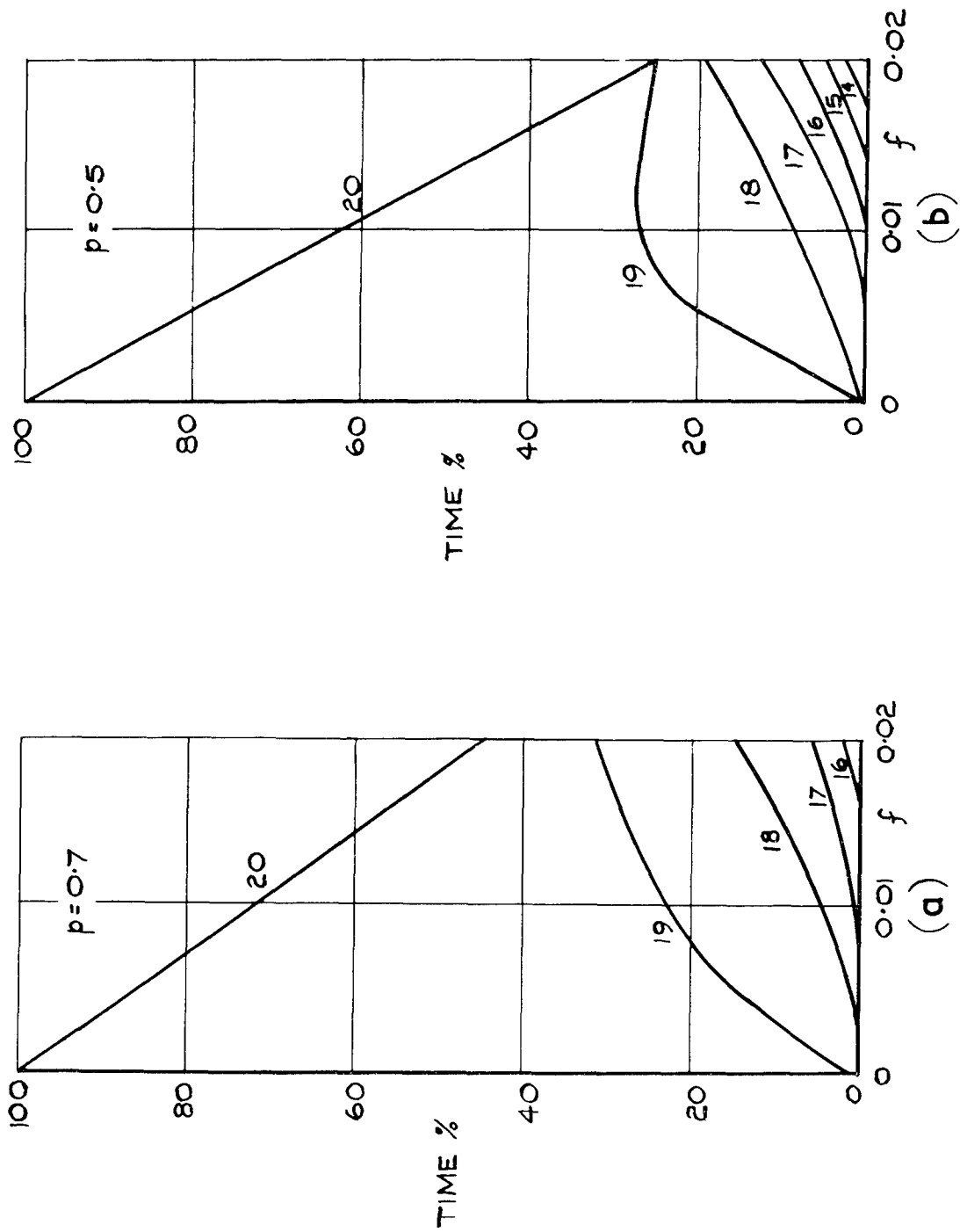


FIG.22 (a&b) STATE DURATIONS AS A FUNCTION OF  $f$  FOR  $N=20$ ,  $p=0.7$  (a) AND  $p=0.5$  (b)



<p>Plimmer, R.N.A.</p> <p>629.195 : 621.396.946</p> <p>ESTABLISHMENT AND MAINTENANCE OF COMMUNICATIONS SATELLITE SYSTEMS (SINGLE LAUNCHES)</p> <p>Royal Aircraft Establishment Technical Report 65152</p> <p>July 1965</p> <p>A theoretical analysis is presented of the number of launches that are required to establish and maintain a satellite communication system. A simple model is used to represent the process in that launches are assumed to take place with a certain a priori probability whilst the satellites have an exponential probability distribution of lifetimes.</p> <p>Graphs and equations are given for the various probability distributions of the system during the establishment and maintenance phases of the system and the effect of the time interval between launches is also examined by using a Markov chain representation of the system. It is noted that many</p> <p>(Over)</p>	<p>Plimmer, R.N.A.</p> <p>629.195 : 621.396.946</p> <p>ESTABLISHMENT AND MAINTENANCE OF COMMUNICATIONS SATELLITE SYSTEMS (SINGLE LAUNCHES)</p> <p>Royal Aircraft Establishment Technical Report 65152</p> <p>July 1965</p> <p>A theoretical analysis is presented of the number of launches that are required to establish and maintain a satellite communication system. A simple model is used to represent the process in that launches are assumed to take place with a certain a priori probability whilst the satellites have an exponential probability distribution of lifetimes.</p> <p>Graphs and equations are given for the various probability distributions of the system during the establishment and maintenance phases of the system and the effect of the time interval between launches is also examined by using a Markov chain representation of the system. It is noted that many</p> <p>(Over)</p>
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